Jacobi identity and intersection of altitudes

It is many years that I know the expression which belongs to V. Arnold and which sounds something like that: "Altitudes (heights) of triangle intersect in one point because of Jacoby identity" or may be even more aggressive: "The geometrical meaning of Jacoby identity is contained in the fact that altitudes of triangle are intersected in the one point". Today preparing exercises for students I suddenly understood a meaning of this sentence. Here it is:

Let \( ABC \) be a triangle. Denote by \( \mathbf{a} \) vector \( BC \), by \( \mathbf{b} \) vector \( CA \) and by \( \mathbf{c} \) vector \( AB: \mathbf{a} + \mathbf{b} + \mathbf{c} = 0 \). Consider vectors \( \mathbf{N}_a = [\mathbf{a}, [\mathbf{b}, \mathbf{c}]], \mathbf{N}_b = [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] \) and \( \mathbf{N}_c = [\mathbf{c}, [\mathbf{a}, \mathbf{b}]]. \) (We denote by \([, ]\) vector product). Vector \( \mathbf{N}_a \) applied at the point \( A \) of the triangle \( ABC \) belongs to the plane of triangle, it is perpendicular to the side \( BC \) of this triangle. Hence the altitude (height) \( h_A \) of the triangle which goes via the vertex \( A \) is egement on the line given by equation \( \mathbf{r}_A(t) = A + t \mathbf{N}_a \). The same is for vectors \( \mathbf{N}_b, \mathbf{N}_c \): Altitude (height) \( h_B \) is on the line which goes via the vertex \( B \) along the vector \( \mathbf{N}_b \) and altitude \( h_C \) (height) is a line which goes via the vertex \( C \) along the vector \( \mathbf{N}_c \). Due to Jacobi identity sum of vectors \( \mathbf{N}_a, \mathbf{N}_b, \mathbf{N}_c \) is equal to zero:

\[
\mathbf{N}_a + \mathbf{N}_b + \mathbf{N}_c = [\mathbf{a}, [\mathbf{b}, \mathbf{c}]] + [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] + [\mathbf{a}, [\mathbf{b}, \mathbf{c}]] = 0 \tag{1}
\]

To see that altitudes \( h_A: A + t \mathbf{N}_a, \ h_B: B + t \mathbf{N}_b \) and \( h_C: C + t \mathbf{N}_c \) intersect at a point it is enough to show that the sum of torques (angular momenta) of vector \( \mathbf{N}_a \) attached at the point \( A \), vector \( \mathbf{N}_b \) attached at the line \( B \), and vector \( \mathbf{N}_c \) attached at the line \( C \) vanishes with respect to some \( M \):

\[
[\mathbf{M}, \mathbf{N}_a] + [\mathbf{M}, \mathbf{N}_b] + [\mathbf{M}, \mathbf{N}_c] = 0. \tag{2}
\]

Indeed it is easy to see that equation (1) implies that relation (2) obeys for an arbitrary point \( M' \) if and only if it obeys for a given point \( M \).

We prove now equation (2) for an arbitrary point \( M \). Denote \( MA = \mathbf{x} \) then using equation (1) we see that for left hand side of the equation (2)

\[
[\mathbf{M}, \mathbf{N}_a] + [\mathbf{M}, \mathbf{N}_b] + [\mathbf{M}, \mathbf{N}_c] = [\mathbf{x}, \mathbf{N}_a] + [\mathbf{x} + \mathbf{c}, \mathbf{N}_b] + [\mathbf{x} + \mathbf{c} + \mathbf{a}, \mathbf{N}_c] =
\]

\[
= [\mathbf{c}, \mathbf{N}_b] + [\mathbf{c} + \mathbf{a}, \mathbf{N}_c] = [\mathbf{c}, [\mathbf{b}, [\mathbf{c}, \mathbf{a}]]] + [\mathbf{c} + \mathbf{a}, [\mathbf{c}, [\mathbf{a}, \mathbf{b}]]] =
\]

\[
[\mathbf{a} + \mathbf{b}, [\mathbf{b}, [\mathbf{a}, \mathbf{b}]]] + [\mathbf{b}, [\mathbf{a} + \mathbf{b}, [\mathbf{a}, \mathbf{b}]]] = (\text{here we used that } \mathbf{a} + \mathbf{b} + \mathbf{c} = 0)
\]

\[
[\mathbf{a}, [\mathbf{b}, [\mathbf{a}, \mathbf{b}]]] + [\mathbf{b}, [\mathbf{a}, [\mathbf{a}, \mathbf{b}]]] + [\mathbf{b}, [\mathbf{b}, [\mathbf{a}, \mathbf{b}]]] = [\mathbf{a}, [\mathbf{b}, [\mathbf{a}, \mathbf{b}]]] + [\mathbf{b}, [\mathbf{a}, [\mathbf{a}, \mathbf{b}]]] =
\]

\[
[a, [b, [a, b]]] + [b, [a, [a, b]]] + [[b, a], [a, b]] - [[b, a], [a, b]] = [a, b], [a, b] = 0.
\]

Jacobi identity

In the last relation we again use Jacobi identity: We see that equation (2) holds, hence altitudes of triangle intersect in one point! Zabavno, da?

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