Jacobi identity and intersection of altitudes

It is many years that I know the expression which belongs to V.Arnold and which sounds something like that: *"Altitudes (heights) of triangle intersect in one point because of Jacoby identity"* or may be even more aggressive: *"The geometrical meaning of Jacoby identity is contained in the fact that altitudes of triangle are intersected in the one point"*. Today preparing exercises for students I suddenly understood a meaning of this sentence. Here it is:

Let ABC be a triangle. Denote by $\mathbf{a}$ vector BC, by $\mathbf{b}$ vector CA and by $\mathbf{c}$ vector AB: $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$. Consider vectors $\mathbf{N_a}=[\mathbf{a},[\mathbf{b},\mathbf{c}]]$, $\mathbf{N_b}=[\mathbf{b},[\mathbf{c},\mathbf{a}]]$ and $\mathbf{N_c}=[\mathbf{c},[\mathbf{a},\mathbf{b}]]$. (We denote by $[\,,\,]$ vector product). Vector $\mathbf{N_a}$ applied at the point A of the triangle ABC belongs to the plane of triangle, it is perpendicular to the side BC of this triangle. Hence the altitude (height) h_A of the triangle which goes via the vertex A is egment on the line given by equation $\mathbf{r}_A(t)=A+t\mathbf{N_a}$. The same is for vectors $\mathbf{N_b},\mathbf{N_c}$: Altitude (height) h_B is on the line which goes via the vertex B along the vector $\mathbf{N}_b$ and altitude h_C (height) is a line which goes via the vertex C along the vector $\mathbf{N}_c$.

Due to Jacobi identity sum of vectors $\mathbf{N_a},\mathbf{N_b},\mathbf{N_c}$ is equal to zero:

$$\mathbf{N_a}+\mathbf{N_b}+\mathbf{N_c}=[\mathbf{a},[\mathbf{b},\mathbf{c}]]+[\mathbf{b},[\mathbf{c},\mathbf{a}]]+[\mathbf{a},[\mathbf{b},\mathbf{c}]]=0 \tag{1}$$

To see that altitudes $h_A\colon A+t\mathbf{N_a}$, $h_B\colon B+t\mathbf{N_b}$ and $h_C\colon C+t\mathbf{N_c}$ intersect at a point it is enough to show that the sum of torques (angular momenta) of vector $\mathbf{N_a}$ attached at the point A, vector $\mathbf{N_b}$ attached at the line B, and vector $\mathbf{N_c}$ attached at the line C vanishes with respect to some M:

$$[MA,\mathbf{N_a}]+[MB,\mathbf{N_b}]+[MC,\mathbf{N_c}]=0\,. \tag{2}$$

Indeed it is easy to see that equation (1) implies that relation (2) obeys for an arbitrary point M' if and only if it obeys for a given point M.

We prove now equation (2) for an arbitrary point M. Denote $MA=\mathbf{x}$ then using equation (1) we see that for left hand side of the equation (2)

$$[MA,\mathbf{N_a}]+[MB,\mathbf{N_b}]+[MC,\mathbf{N_c}]=[\mathbf{x},\mathbf{N_a}]+[\mathbf{x}+\mathbf{c},\mathbf{N_b}]+[\mathbf{x}+\mathbf{c}+\mathbf{a},\mathbf{N_c}]=$$

$$=[\mathbf{c},\mathbf{N_b}]+[\mathbf{c}+\mathbf{a},\mathbf{N_c}]=[\mathbf{c},[\mathbf{b},[\mathbf{c},\mathbf{a}]]]+[\mathbf{c}+\mathbf{a},[\mathbf{c},[\mathbf{a},\mathbf{b}]]]=$$

$$[\mathbf{a}+\mathbf{b},[\mathbf{b},[\mathbf{a}+\mathbf{b},\mathbf{a}]]]+[\mathbf{b},[\mathbf{a}+\mathbf{b},[\mathbf{a},\mathbf{b}]]]=\ \text{(here we used that } \mathbf{a}+\mathbf{b}+\mathbf{c}=0)$$

$$[\mathbf{a},[\mathbf{b},[\mathbf{b},\mathbf{a}]]]+[\mathbf{b},[\mathbf{b},[\mathbf{b},\mathbf{a}]]]+[\mathbf{b},[\mathbf{a},[\mathbf{a},\mathbf{b}]]]+[\mathbf{b},[\mathbf{b},[\mathbf{a},\mathbf{b}]]]=[\mathbf{a},[\mathbf{b},[\mathbf{b},\mathbf{a}]]]+[\mathbf{b},[\mathbf{a},[\mathbf{a},\mathbf{b}]]]=$$

$$\underbrace{[\mathbf{a},[\mathbf{b},[\mathbf{b},\mathbf{a}]]]+[\mathbf{b},[[\mathbf{b},\mathbf{a}]]]+[[\mathbf{b},\mathbf{a}],[\mathbf{a},\mathbf{b}]]}_{\text{Jacobi identity}}-[[\mathbf{b},\mathbf{a}],[\mathbf{a},\mathbf{b}]]=[[\mathbf{a},\mathbf{b}],[\mathbf{a},\mathbf{b}]]=0\,.$$

In the last relation we again use Jacobi identity: We see that equation (2) holds, hence altitudes of triangle intersect in one point! Zabavno, da? ■

Hovik Khudaverdian (24.01.07)