The arithmetico-geometrical mean,
the potential of circle and elliptic functions.

§0 The arithmetico-geometrical mean. What is it.

Let $M(a,b)$ be the arithmetico-geometrical mean of two positive numbers a, b, i.e. for

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where $a_0 = a, b_0 = b$ and $a_{n+1} = \frac{a_n+b_n}{2}$ (arithmetic mean) and $b_{n+1} = \sqrt{a_n b_n}$ (geometric mean. Since geometric mean is less or equal to arithemtic mean this limit is well defined.

Examples

a	b
10	1
5, 5	3, 16227766
4, 33113883	4, 170434885
4, 250786858	4, 250027349
4, 250407103	4, 250407086
4, 250407095	4, 250407095
4, 250407095	4, 250407095
...	...

a	b
100	10
55	31, 6227766
43, 3113883	41, 70434885
42, 50786858	42, 50027349
42, 50407103	42, 50407086
42, 50407095	42, 50407095
42, 50407095	42, 50407095
42, 50407095	42, 50407095
...	...

a	b
18	8
13	12
12, 5	12, 489996
12, 494998	12, 494997
12, 4949975	12, 4949975
12, 4949975	12, 4949975
...	...

(0)

We see that after four-five iterations we come to an answer up to 10^{-8}.

§1 Integral representation of the arithmetico-geometrical mean

For the arithmetico-geometric mean there is a beautiful representation through the integral: Consider the following integral

$$S(a,b) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(a^2\cos^2 t + b^2 \sin^2 t)}} dt\,. \tag{1}$$

Lemma(Gauß) The integral $S(a,b)$ does not change if $(a,b) \to (a_1, b_1)$ where a_1 is arithmetic mean of a, b and b_1 is geometric mean of a, b:

$$S(a,b) = S\left(\frac{a+b}{2}, \sqrt{ab}\right), \quad (a, b \geq 0) \tag{1a}$$

Thus this lemma implies that

$$S(a,b) = \lim S(a_n, b_n) = S(M(a,b).M(a,b)) = \frac{\pi}{2M(a,b)}\,.$$

We come to the following formula

$$\frac{\pi}{2M(a,b)} = S(a,b) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(a^2\cos^2 t + b^2\sin^2 t)}} dt$$

or

$$M(a,b) = \frac{2}{\pi \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(a^2\cos^2 t + b^2 \sin^2 t)}} dt} \,. \tag{1c}$$

We see that the formula (1c) gives the nice and short way to calculate numerically the integral (1) since calculation of $M(a,b)$ is very convergent (see example (0) above). This is why Gauß came to this notion.

Example Calculate *potential of the circle.**. Consider circle of the radius R with charge Q in the plane. Then potential on the distance ρ (in the plane) from the centre equals to

$$U(\rho) = \frac{Q}{2\pi} \int_0^{2\pi} \frac{dt}{\sqrt{\rho^2 + R^2 - 2R\rho \cos t}}$$

One can see that **

$$\rho^2 + R^2 - 2R\rho\cos t = \rho^2 + R^2 - 2R\rho\left(2\cos^2\frac{t}{2} - 1\right) = (\rho+R)^2 \sin^2\frac{t}{2} + (\rho - R)^2 \cos^2\frac{t}{2}$$

and

$$U(\rho) = \frac{Q}{2\pi} \int_0^{2\pi} \frac{dt}{\sqrt{\rho^2 + R^2 - 2R\rho \cos t}} =$$

$$\frac{Q}{\pi} \int_0^{\pi} \frac{dt}{\sqrt{((\rho+R)^2\cos^2 t + (\rho-R)^2 \sin^2 t)}} = \frac{2Q}{\pi} S(\rho+R, \rho-R) = \frac{Q}{M(R+\rho, R-\rho)} \,.$$

We come to very beautiful

Fact(**Gauß**) Let the charge Q is homogeneously distributed on the circle of the radius R on the plane. Then the potential at the point of the plane on the distance ρ is equal to

$$U(\rho) = \frac{Q}{M(\rho+R, \rho-R)}$$

§2 *Proof of Gauss lemma*

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Now prove the Gauß lemma (1a).

First note that under substitution $x = \sin t$ we come to

$$S(a,b) = \int_0^{\frac{\pi}{2}} \frac{ab}{\sqrt{(a^2\cos^2 t + b^2 \sin^2 t)}} dt = \frac{1}{a} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1 - \frac{a^2-b^2}{a^2}x^2}} \,. \tag{2}$$

* Calculation of this integral was the reason why Gauß introduced this notion (?)

** canonical way of calculating potential it is an expansion on Legendre polynomials.

Here we see so called complete elliptic integral of the first kind:

$$K(k)=\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \tag{2a}$$

and we have that

$$S(a,b)=\frac{1}{a}K\left(\frac{\sqrt{a^2-b^2}}{a}\right)\,. \tag{2b}$$

Remark Sure we need only complete elliptic integral for calculating (1) but to understand deeper the symmetries which imply these properties we need also also the function, so called Jacobi first kind elliptic integral:

$$u=\int_0^X \frac{dt}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,,\qquad \text{where } X=\mathrm{sn}\,(u;k)\,. \tag{2c}$$

Thus we come to elliptic function, "elliptic sinus"

Rewrite the statement of lemma $S(a,b)=S(\frac{a+b}{2},\sqrt{ab})$ in terms of function $K(k)$:

$$\frac{1}{a}K\left(\frac{\sqrt{a^2-b^2}}{a}\right)=S(a,b)=S\left(\frac{a+b}{2}\right)=\frac{2}{a+b}K\left(\frac{\sqrt{\left(\frac{a+b}{2}\right)^2-\left(\sqrt{ab}\right)^2}}{\frac{a+b}{2}}\right)=\frac{2}{a+b}K\left(\frac{a-b}{a+b}\right)$$

If we put $k=\frac{a-b}{a+b}$ then $b=\frac{1-k}{1+k}a$ and $\frac{\sqrt{a^2-b^2}}{a}=\frac{2\sqrt{k}}{1+k}$. Thus the relation $S(a,b)=S\left(\frac{a+b}{2},\sqrt{ab}\right)$ is equivalent to the relation

$$K(k)=K\left(\frac{a-b}{a+b}\right)=\frac{a+b}{2a}K\left(\frac{\sqrt{a^2-b^2}}{a}\right)=\frac{1}{1+k}K\left(\frac{2\sqrt{k}}{1+k}\right)$$

Now we will proof the relation

$$K(k)=\frac{1}{1+k}K\left(\frac{2\sqrt{k}}{1+k}\right) \tag{3}$$

which is equivalent to the relation $S(a,b)=S\left(\frac{a+b}{2},\sqrt{ab}\right)$.

Left and right hand sides are integrals: We have to prove that

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{1}{1+k}\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k'^2y^2)}} \tag{3a}$$

We will prove it if we will find the anzats $y = F(x)$ such that

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{1+k}\frac{dy}{\sqrt{(1-y^2)(1-k'^2y^2)}}, \quad \left(k' = \frac{2\sqrt{k}}{1+k}\right) \tag{3b}$$

and $F(0) = 0, F(1) = 1^*$ In this case we will immediately will come to the relation (3a).

Observation The following fraction

$$y = \frac{(1+k)x}{1+kx^2} \tag{3c}$$

is the suitable anzats. (In other words the ansatz (3a) is the integral of differential equation (3b) with initial condition $y(0) = 0$).

Proof. Direct calculations: We have that $dy = \frac{(k+1)(1-kt^2)}{(1+kx^2)^2}dx$ and

$$1 - y^2 = 1 - \frac{(1+k)^2x^2}{(1+kx^2)^2} = \frac{(1+kx^2)^2 - (1+k)^2x^2}{(1+kx^2)^2} = \frac{(1-x^2)(1-k^2x^2)}{(1+kx^2)^2}$$

Then

$$\frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} = \left(\frac{dx}{dy}dy\right)\left(\frac{1}{(1+k^2x^2)\sqrt{1-y^2}}\right) =$$

$$\left(\frac{(1+kx^2)^2}{(k+1)(1-kx^2)}dy\right)\left(\frac{1}{(1+k^2x^2)\sqrt{1-y^2}}\right) = \frac{dy}{y\sqrt{1-y^2}}\frac{x}{1-kx^2}$$

We have that $\frac{x}{1-kx^2} = \frac{y}{\sqrt{(1+k)^2-4ky^2}}$ (To see it note that $z = \left(\frac{x}{1-kx^2}\right)^2$ is rational function on y: $z = \left(\frac{t}{1-kx^2}\right)^2 = \frac{x^2}{(1+kx^2)^2-4kx^2} = \frac{1}{(1+k)^2y-4ky}$)). This leads to the relation (3b).

Sure this is “rabbit from the hat” proof. The mistery , the “raison d‘etre” of the anzats (3a) can be explained by the theory of elliptic functions †.

We try to explain the phenomenon on the following toy example:

§4 Toy example: trigonometric functions instead elliptic.

Consider the following toy -analogue of the Observation (3):

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* The proof that it is equal to elliptic integral is very good demonstration of so called Zagier-Kontsevitch periods (Kontsevitch and Zagier conjectured that two periods (integrals) are equal if it can be checked by standard operations.).

† There is one beautiful anzats: $\varphi\colon \tan\varphi = \frac{b}{a}\tan t$. Then one can easy check that $\frac{dt}{\sqrt{(a^2\cos^2 t + b^2\sin^2 t)}} = \frac{d\varphi}{\sqrt{(b^2\cos^2\varphi + a^2\sin^2\varphi)}}$, i.e. integrand is invariant under changing $t \mapsto \varphi$, $a \to b$, $b \to a$. But this is ”zamechanije v bok”

Toy-observation Consider differential equation:

$$\frac{dx}{\sqrt{1-x^2}} = M\frac{dy}{\sqrt{1-y^2}}$$

where M is a parameter. Try to find integrals for this equation.

E.g. for $M=2$ $x = 2y\sqrt{1-y^2}$. For $M=3$ $x = 3y-4y^3$,...

We cannot find non-trivial integral of motion for $M=\sqrt{2}$. Here the reason is obvious: The differential equation $\frac{dx}{\sqrt{1-x^2}} = M\frac{dy}{\sqrt{1-y^2}}$ means that

$$\arcsin x = M \arccos y + C$$

Function $\arcsin x$ is multi-valued function which is defined up to $\pm 2\pi k$ and function $M\arccos y$ is defined up to $\pm 2\pi M k$

We see that the differential equation $\frac{dx}{\sqrt{1-x^2}} = M\frac{dy}{\sqrt{1-y^2}}$ is integrable if M is rational.

In the case of elliptif functions (3b)-like differential equation

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \dots\frac{dy}{\sqrt{(1-y^2)(1-k'^2y^2)}},$$

if parallelograms of periods for elliptic are related with rational transformations.

In particular Gauß transformation corresponds to transformatioon $(\omega_1,\omega_2)\to(\omega_1,2\omega_2)$. The correspondi In the case of elliptic functions the Gauß anzats leads to the follwoing relation between elliptic functions:

$$u = \int_0^X \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{1+k}\int_0^{Y(X)} \frac{dy}{\sqrt{(1-y^2)(1-k'^2y^2)}},$$

with

$$Y(X) = \frac{(1+k)X}{1+kX^2}, \text{and } k' = \frac{2\sqrt{k}}{1+k}$$

(If $X=1$ then $Y=1$ and we come to the relation (3)).

Now we have that for elliptic function $\mathrm{sn}\,(x;k)$ (see 2(c)) the following identity:

$$\frac{sn\,((k+1)u;k')}{k+1} = \frac{sn\,(u;k)}{1+k\mathrm{sn}^{\,2}(u;k)}$$

This is the relation between elliptic functions with periods (K,iK') and $(K,2iK')$.

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