Locally Euclidean Geometries and hyperbolic geometry

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Abstract

Geometry on the surface of the cylinder is locally Euclidean. An "ant-mathematician" who lives on the cylinder will not distinguish the geometry of the surface at small distances from the Euclidean geometry; the Pythagorean Theorem will be almost the same, and for "not too large" triangles the sum of the angles will be π. In the first part of talk, we will study locally Euclidean two-dimensional geometries. We will study these geometries by using discrete subgroups of the isometry group of the Euclidean plane E^2. The list of locally Euclidean geometries is exhausted by the geometries on the surface of the cylinder, on the surface of the torus, on the surface of the "twisted cylinder" (the Moebius band), and on the so-called Klein bottle. In the second part of the talk, we will consider the set of locally Euclidean geometries, and will show that this set can be naturally parametrized by the points of the Lobachevsky (hyperbolic) plane.

Contents

1 Locally Euclidean surfaces

1.1 (Uniformly) locally Eucldean surfaces

We consider locally Euclidean 2-dimensional geometries. An arbitrary 2-dimensional geometry can be considered as 2-dimensional Riemannian surface, (M, G). M is a surface, and G defines scalar product of tangent vectors, i.e. length of an arbitrary curve. For arbitrary curve $\mathbf{x} = \mathbf{x}(t), t_1 \leq t \leq t_2$

$$\text{length of the curve} = \int_{t_1}^{t_2} \sqrt{(\mathbf{v}(t), \mathbf{v}(t))} dt \,, \text{ scalar product } (\mathbf{v}(t), \mathbf{v}(t)) = G(\mathbf{v}(t), \mathbf{v}(t)) \,,$$

where $\mathbf{v}(t)$ is velocity vector. In local coordinates x^i, the curve has appearance $\mathbf{x}(t) = x^i(t)$, $G = g_{ik}(x)dx^i dx^k$, $\mathbf{v}(t) = v^i(x(t))\partial_i$, and the scalar product of velocity vector on itself is equal to

$$(\mathbf{v}, \mathbf{v}) = v^i(x(t))g_{ik}(x(t))v^k(x(t)) = \frac{dx^i(x(t))}{dt} g_{ik}(x(t)) \frac{dx^k(x(t))}{dt} \,,$$

i.e.

$$\text{length of the curve} = \int_{t_1}^{t_2} \sqrt{(\mathbf{v}(t), \mathbf{v}(t))} dt = \int_{t_1}^{t_2} \sqrt{\frac{dx^i(x(t))}{dt} g_{ik}(x(t)) \frac{dx^k(x(t))}{dt}} .dt$$

We consider surfaces which locally look as Euclidean plane.

We say that M is uniformly locally Eucldean if

- in a vicinity of arbitrary points there exist Eucldean coordinates, i.e. the coordinates u, v such that $G = du^2 + dv^2$ in these coordinates.

- this neighborhood is enough large: there exists $r > 0$ such that in a vicinity of arbitrary point there exist Eucldiean coordinates u, v which are defined at least in the circle of radius $\geq r$.

Exercise Show that the surface of cylindre is locally Eucldean.

Exercise Show that the surface of sphere is not locally Eucldean.

Exercise Show that domain $a < x < b$ of $\mathbf{E}^2$ is locally Eucldean but it is not uniformly locally Eucldean, and compare it with surface of cylindre, (which is uniformly locally Euclidean.)

(We suppose that metric on the surface M in $\mathbf{E}^3$ is the metric induced from $\mathbf{E}^3$.)

In this talk we will consider only uniformly locally Euclidean surface, and we will call them sometimes just locally Eucldean.

1.2 Examples of locally Euclidean surfaces and subgroups of $\mathbf{E}(2)$.

We considered surface of cylindre. It is uniformly locally Euclidean surface. How to come to another examples. Here we suggest the regular way to come to all such examples. It is in the spirit of Klein Erlangen programme.

1.2.1 Subgroups of group $\mathbf{E}(2)$ and surfaces

Let Γ be an arbitrary subgroup of group of isometries of $\mathbf{E}^2$.

Assign to the group Γ a space M_Γ of orbits of Γ-group action on $\mathbf{E}^2$, $M_\Gamma = \mathbf{E}^2 \backslash \Gamma$.

We denote the points of the space M_Γ, by handwriting letters $\mathcal{A}, \mathcal{B}, C, \mathcal{D}, \ldots$. These points are orbits of group Γ action. Every point $A \in \mathbf{E}^2$ produces the point $\mathcal{A} \in \mathbf{E}^2 \backslash \Gamma$, the equivalence class of a point A with respect to the group Γ: $\mathcal{A} = [A]_\Gamma\,, \qquad g \in \Gamma\,, A' = g(A) \in [A]$.

To establish the geometry on M we define the distance between points $\mathcal{A}, \mathcal{B}$ as the minimal distance between the orbit $\{A^g\}$ and $\{B^g\}$:

$$\text{if } \mathcal{A} = [A] \text{ and } \mathcal{B} = [B] \text{ then } d(\mathcal{A}, \mathcal{B}) = \min_{g, g' \in \Gamma} d(A^g, B^{g'})$$

Exercise 1 Let $\mathbf{a} \neq 0$ be an arbitrary vector in $\mathbf{E}^2$ and $\Gamma = \{T_{n\mathbf{a}}\}$ be a group of translations generated by the translation on vector $\mathbf{a}$:

$$T_{n\mathbf{a}} \colon \mathbf{r} \to \mathbf{r} + n\mathbf{a} \,.$$

Describe the geometry M_Γ and show that this is geometry of cylinder.

Exercise 2 Let $\Gamma = C_2$ be group of reflection with respect to the line l.

Describe the geometry M_Γ and show that this is not uniformly locally Euclidean manifold.

1.2.2 Uniformly discontinuous subgroups of $\mathbf{E}(2)$

We say that the subgroup Γ acts properly discontinuous on $\mathbf{E}^2$ if there exists δ such that for an arbitrary point $A \in \mathbf{E}^2$, and for an arbitrary non-identity element $g \in \Gamma$

$$d(A, g(A)) \geq \delta \,.$$

In other words it means that the distance between distinct points of an arbitrary orbit exceeds the δ.

Why these groups are interesting? Because every such a group defines locally Eucldean manifold[1].

Proposition 1. *If action of group Γ has a fixed point (there exists $\mathbf{r}_0 \in \mathbf{E}^2$ such that for all $g \in \Gamma$, $g(\mathbf{r}_0) = \mathbf{r}_0$) (il faut dire mieux), then M_Γ is not uniformly locally Euclidean.*

Proof. If $A = \mathbf{r}_0$ is a fixed point, then for arbitrary $g \neq 1$ $d(A, A^g) \geq \delta$, and on the other hand $d(A, A^g) = 0$, if Γ acts uniformly discontinuou. Contradiction. □

Proposition 2. *If $\Gamma \in \mathbf{E}(2)$ is uniformly discontinuous group, then the surface $M_\Gamma = \mathbf{E}^2 \backslash \Gamma$ is uniformly local.*

Proof. .

Let Γ be uniformly discontinuous, i.e. there exists $\delta > 0$ such that for an arbitrary $g \in \Gamma$ and for an arbitrary point $A \in \mathbf{E}^2$ $d(A^g, A) < \delta \Rightarrow g = 1$. Let B be an arbitrary point which belongs to the disc $D_{\frac{\delta}{2}}(A)$ Consider orbits

[1] In fact the inverse implication is true also, and the set of all (uniformly) locally Euclidean surfaces is in one-one correspondence with the set of all uniformly discontinuous subgroups of group $\mathbf{E}^2$.(see further)

$\mathcal{A}$ and $\mathcal{B}$ of these points. It is easy to see from triangle inequality that for arbitrary points $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ the distance between these points is bigger or equal to $r = \frac{\delta}{2}$. Indeed let $A" = A^g$, and $B' = B^h$. Denote $\tilde{B} = B^{hg^{-1}}$. Then $d(A', B') = d\left(A, \left(B^h\right)^{g^{-1}}\right) = d\left(A, \tilde{B}\right)$, and by triangle inequality

$$d(A', B') = d(A, \tilde{B}) \geq \left| d(B, \tilde{B}) - d(A, B) \right| > \delta$$

if $B \neq \tilde{B}$. Thus we see that in the case if two points A and B are closer than $\frac{\delta}{2}$, then the distance between orbits $\mathcal{A}$ and $\mathcal{B}$ is equal to the distance $d(A, B)$. This implies that M_Γ is uniformly locally Euclidean if Γ is uniformly discontinous.

□

1.2.3 Classiication of all uniformly discontinuous subgroups of $\mathbf{E}(2)$

First of all recall the classic Theorem:

Theorem 1. *(Chazles+?) Any isometry of* $\mathbf{E}^2$ *is rotation, or translation or glided reflection.*

This Theorem possesses two statements. First that an arbitrary (even non-linear map) which is isometry has appearance

$$F(\mathbf{r}) = A\mathbf{r} + \mathbf{b}\,,$$

where A is linear operator, and the second statement that this linear map is rotation (with respect to some centre) or translation or glided reflection.

We can prove the first statement under the assumption that $F(\mathbf{r})$ is smooth map of $\mathbf{E}^2$ in $\mathbf{E}^2$.

Suppose now that $F(\mathbf{r}) = A(\mathbf{r}) + \mathbf{b}$.

I-st case) orientation is preserved, i.e. $\det A = 1$. If $A = 1$ then it is translation, if $A \neq 1$ then operator $A - 1$ is invertible, and

$$\mathbf{r} = A(\mathbf{r}) + \mathbf{b} = \mathbf{a} + (A\,(\mathbf{r} - \mathbf{a}))\ \text{, where }, \mathbf{a} = (1 - A)^{-1}(\mathbf{b})\,,$$

i.e. this affine transformation is a rotation around the point $O - \mathbf{a}$.

II-nd case) orientation is not preserved, i.e. $\det A = -1$. This operator has eigenvector $\mathbf{n}$

One can see that we come to reflection with respect to the line along the vector $\mathbf{n}$ and translation along $\mathbf{n}$, i.e. glided reflection.

Now using this Theorem classify uniformly discontinuous subgroups of isometry group.

Let Γ be a subgroup of $\mathbf{E}(2)$ which acts on $\mathbf{E}^2$ uniformly discontinuous, i.e. there exists $\delta > 0$ such that for an arbitrary point $A \in \mathbf{E}^2$ and an arbitrary $g \in \Gamma$

$$d(A, A^g) < \delta \Rightarrow g = 1 \,.$$

It follows from proposition 1 that the subgroup Γ contains only translations and non-trivial glide reflections.

If group $\Gamma = e$ this is trivial: $M_\Gamma = \mathbf{E}^2$.

Denote by $\Gamma^{(0)}$ the subgroup of Γ which preserve orientation, i.e. subgroup of translations.

Proposition 3. *The subgroup $\Gamma^{(0)}$ of uniformly discontinuous grroup Γ of orientation preserving transformations is*

- *the group of translations generated by arbitrary non-zero vector* $\mathbf{a}$

$$\Gamma^{(0)} = \Gamma^{(0)}0_{\mathbf{a}} = \{T_{n\mathbf{a}} \colon T_{n\mathbf{a}}(\mathbf{r}) = \mathbf{r} + n\mathbf{a}, \text{ ,where } n = 0, \pm 1, \pm 2, \dots\}$$

- *the group of translations generated by arbitrary two non-zero linearly independent vectors* $\mathbf{a}, \mathbf{b}$

$$\Gamma^{0)} = \Gamma^{(0)}_{\mathbf{a},\mathbf{b}} = \{T_{m\mathbf{a}+n\mathbf{b}} \colon T_{m\mathbf{a}+n\mathbf{b}}(\mathbf{r}) = \mathbf{r} + m\mathbf{a} + n\mathbf{b}, \text{ ,where } m, n = 0, \pm 1, \pm 2, \dots\}$$

On the base of this proposition study the general case.

Let Γ be an arbitrary uniformly discontinuous subgroup of $\mathbf{E}(2)$. Then $\Gamma^{(0)} = \Gamma^{(0)}_{\mathbf{a}}$ or $\Gamma^{(0)} = \Gamma^{(0)}_{\mathbf{a},\mathbf{b}}$.

`First case` $\Gamma^{(0)} = \Gamma^{(0)}_{\mathbf{a}}$.

if $\Gamma = \Gamma_0 = \{T_{n\mathbf{a}}\}$, and $M = \mathbf{E}^2 \backslash \Gamma$ is cylindre.

Now suppose $\Gamma \neq \Gamma_0$, i.e. Γ possesses glide reflections. Let $S = S_{l,\mathbf{b}} \in \Gamma$ (reflection with respect to the line l directed along the vector $\mathbf{b}$ and translation on the vector $\mathbf{b}$). Then $S^2 = T_{2\mathbf{b}}$. Hence $\mathbf{b} = \frac{k\mathbf{a}}{2}$ for some integer k. This integer k has to be odd, since if $p = 2p$ then the transformation $T_{-p\mathbf{a}}S$ is the reflection, and it possesses fixed point (see the porposition 1). We see that in this case G is generated by translation $T_{\mathbf{a}}$ and glided reflection $S_{l,\frac{\mathbf{a}}{2}}$

In this case $M = \mathbf{E}^2 \backslash \Gamma$ is twisted cylindre, (Mobius strip with infinite sides)

`Second case` $\Gamma^{(0)} = \Gamma^{(0)}_{\mathbf{a},\mathbf{b}}$.

Let $\Gamma_0 = \{T_{n\mathbf{a}}\}$. Study again two possibilites.

First: $\Gamma = \Gamma_0 = \{T_{n\mathbf{a}}\}$. In this case

$M = \mathbf{E}^2 \backslash \Gamma$ is a torus.

Suppose now that $\Gamma \neq \Gamma_0$ and it possesses glided reflections. One can see that this can happen only if vectors **a** and **b** are orthogonal to each other. We will come to $M = \mathbf{E}^2 \backslash \Gamma$ is a Klein bottle.

We come to the theorem classifying all the locally Euclidean surfaces corresponding to uniformly discontinuous isometry subgroups.

Theorem 2. *Let Γ be an arbitrary uniformly discontinuous subgroup of isometries group of $\mathbf{E}^2$. Then the following possibilities may occur (ca il faut dire mieux!!!)*

- *I-st case (trivial)*

  $\Gamma = e$ *has only identity element. Then* $M = M \backslash \Gamma = \mathbf{E}^2$

- *II-nd case*

  Group $\Gamma = \{T_{m\mathbf{a}}\}$ *is generated by translation on vector* **a**, *where* **a** *is an arbitrary non-zero vector. Then* $M = M \backslash \Gamma$ *is cylindre*

- *III-rd case*

  Group $\Gamma \colon \Gamma^{(0)} = \{T_{n\mathbf{a}}\}$, *however,* $\Gamma \neq \Gamma^{(0)}$. *This group is generated by translation on vector* **a**, *and glide reflection* $S_{l,\frac{\mathbf{a}}{2}}$, *where the line* l *goes along the vector* **a** *Then* $M = M \backslash \Gamma$ *is twisted cylindre (Mobius).*

- *IV-th case*

  Group $\Gamma = \{T_{m\mathbf{a}+\mathbf{n}\mathbf{b}}\}$ *is generated by translation on vectors* **a** *and* **b**, *where* $\mathbf{a}, \mathbf{b}$ *are arbitrary linearly independent vectors. Then* $M = M \backslash \Gamma$ *is a torus*

- *V-th case*

  Group is generated by translation on vectors **a** *and* **b**, *and glided reflection* $S_{l,\frac{\mathbf{a}}{2}}$, *where the line* l *goes along the vector* **a** *where* $\mathbf{a}, \mathbf{b}$ *are arbitrary non-zero vectors which are orthogonal! to each other. Then* $M = M \backslash \Gamma$ *is a Klein bottle.*

1.3 The final classification

We classified in the Theorem above all locally Euclidean surfaces which are generated by uniformly discontinuous groups.

Is an arbitrary uniformly locally Euclidean surface M generated by uniformluy discontinuous group? Yes!

To prove this statement we consider the following construction.

Take an arbitrary point $A \in \mathbf{E}^2$ and $\mathcal{A} \in M$. Choose arbitrary Cartesian coordinates x, y in $\mathbf{E}^2$ adjusted to the point A ($x_A = y_A = 0$ and choose arbitrary local Euclidean coordinates $\{u_0, v_0\}$ on M in vicinity of point $\mathcal{A}$ ($u_M = v_M = 0$). Thus we define a map $\mathbf{E}^2 \to M$ for an arbitrary point $\mathbf{r} \in \mathbf{E}^2$ which is in the disc $D_r(A)$. If point is on the distance bigger than r but less than $2r$ we can consider two discs, e.t.c....

This we will construct a map

$$\begin{array}{c} \mathbf{E}^2 \\ p \downarrow \\ M \end{array}$$

such that $p(A) = \mathcal{A}$. Taking the preimage of arbitrary point $\mathcal{B} \in M$ one come to the set of points which define the uniformly discontinuous group.

We come to

Theorem 3. *Let M be an arbitrary uniformly locally Euclidean surface. Then there exists uniformly discontinuous group Γ such that*

$$M = M_\Gamma$$

It follows from this Theorem and Proposition 3 the following corollary

Corollary 1. *Let M be an arbitrary uniformly locally Eucldean surface. Then $M = \mathbf{E}^2$ or M is cylindre, or M is twisted cylidnre, or M is torus, or M is Klein bottle.*

2 Space of locally Eucldean geometries

Geometries on cylindres are similar to each other, the same about geometries of twisted cylindres and Klein bottles.

Consider the space of geometries on tori.

Every lattice $T_{\mathbf{a},\mathbf{b}}$ defines geometry $M_{\mathbf{a},b} = M \backslash \{T_{m\mathbf{a}+n\mathbf{b}}\}$.

Questions arises.

At what extend lattices depend on vectors $\mathbf{a}, b$. At what extend geometries (tori) $M_{\mathbf{a},\mathbf{b}}$ depend on lattices?

What is a geometry in the space of geometries on tori?

2.1 Lattices— Geometries on tori—unimodular group

Definition 1. Consider pair of arbitrary lattices $T_{\mathbf{a},b}$ and $T_{\mathbf{a}',\mathbf{b}'}$ and corresponding tori $M_{\mathbf{a},\mathbf{b}}$ and $M_{\mathbf{a}',\mathbf{b}'}$.

Two geometries $M_{\mathbf{a},\mathbf{b}}$ and $M_{\mathbf{a}',\mathbf{b}'}$ are the same if

$$\text{lattices } T_{\mathbf{a},b} \text{ and } T_{\mathbf{a}',\mathbf{b}'} \text{ coincide}\,. \tag{2.1}$$

Two geometries are the same if

$$\text{there exists isometry } F \text{ such that } \mathbf{a}' = F(\mathbf{a}) \text{ and } \mathbf{b}' = F(\mathbf{b})\,. \tag{2.2}$$

Two geometries $M_{\mathbf{a},b}$ and $M_{\mathbf{a}',b'}$ are similar if

$$\mathbf{a}' = \lambda\mathbf{a}\,, \mathbf{b}' = \lambda\mathbf{b}\,. \tag{2.3}$$

The first and second conditions are obvious, the condition is natural. They lead to very amazing consequences.

Proposition 4. *Two lattices $T_{\mathbf{a},\mathbf{b}}$ and $T_{\mathbf{a}',\mathbf{b}'}$ coincide if and only if*

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$$

such that

$$ps - qr = \pm 1$$

and

$$p, q, r, s \quad \textit{are integers}$$

in other words if these lattices are related by unimodular transformation in integers.

The proof is evident, and it is very illuminating tpo do the following exercises.

Exercise Check that lattices

$$T_{\mathbf{a},\mathbf{b}} \quad \text{and} \quad T_{41\mathbf{a}+5\mathbf{b},8\mathbf{a}+\mathbf{b}}$$

are the same in spite of the fact that parallelograms $\Pi_{\mathbf{a},\mathbf{b}}$ and $\Pi_{41\mathbf{a}+5\mathbf{b},8\mathbf{a}+\mathbf{b}}$ look very different!

Exercise Prove that lattices $T_{\mathbf{a},\mathbf{b}}$ and $T_{\mathbf{a}',\mathbf{b}}$ coincide if and only if parallelograms $\Pi_{\mathbf{a},\mathbf{b}}$ and $\Pi_{\mathbf{a}',\mathbf{b}'}$ have the same area.

Now identify all similar tori and tori which are related with each other by unimodular transfromation, and find distance between geometries.

Let $T_{\mathbf{a},\mathbf{b}}$ be a lattice.

Assign to this lattice the complex number

$$z = \frac{a_x + ia_y}{b_x + ib_y}$$

in the case if $\mathbf{a} \times \mathbf{b}$ is poisitve, and the inverse complex number in the case if $\mathbf{a} \times \mathbf{b}$ is negative.

Thus using equations (2.2) and (2.3) we come to

Proposition 5. *Set of all lattices is in one-one correspondence with points of upper half-plane* $\mathbf{H}$.

It follows from (2.1) the following fundamental statement

Theorem 4. *Set of all lattices is the set of all points of upper half-plane, and two points define the same geometry if they are related with transformation*

$$z' = \frac{pz+q}{rz+s}\,, \text{with}\, ps - qr = 1\,, \quad \text{or}\ z' = \frac{p\bar{z}+q}{r\bar{z}+s}\,, \text{with}\, ps - qr = -1\,, \quad (2.4)$$

where

$$p,q,r,s, \text{ are integers} \quad (2.5)$$

i.e. they are related with transformation of group $PSL_2(\mathbf{Z})$ and its conjugate.

Notice that we come to Lobachevsky (hyperbolic) plane [2].

One can see using equation (2.1) that the following statement holds:

[2] If we omit the condition (2.5) we come to the group (2.4) of all holomorphic bijections $\mathbf{H} \leftrightarrow \mathbf{H}$.

Theorem 5. *The set of all geometries on tori can be parameterised by the points of Lobachevsky (hyperbolic) plane. Two geometries are similar if and only if they are related by transformation* (2.4)*, i.e. transformation of group* $PSL(2, R) + \text{conjugate}$.

2.2 Geodesics and Metric of hyperbolic geometry

Group of isometries defines geodesics and metrics (up to proportionality) of hyperbolic geometry

We know that ransformations

$$z' = \frac{pz+q}{rz+s}\,, \text{with}\, ps - qr > 0\,, \quad \text{or } z' = \frac{p\bar{z}+q}{r\bar{z}+s}\,, \text{with}\, ps - qr < 1\,, \quad (2.6)$$

i.e. transformations (2.4) without condition (2.5)[3] are isometries of this geometry.

2.2.1 Hyperbolic isometries

Exercise Show that translations are hyperbolic isometries

Exercise Show that dilations are hyperbolic isometries

Exercise Show that inversion with centre at $y = 0$ is hyperbolic isometry.

Exercise Show that arbitrary hyperbolic isometry is composition of these transformations.

2.2.2 Geodesics as locus of fixed points

Lemma 1. *Let a curve C be a locus of fixed points of an isometry. Then this curve is geodesic.*

Exercise Prove this lemma

Using this lemma and exercises in 2.2.1 one can see that an arbitrary upper-half circle with the centre at $y = 0$ and vertical lines are geodesics.

[3]One can see that this is equivalent (2.4) in spite of the fact that we did not put the condition of unimodularity

2.2.3 Formula for distance

A distance on $\mathbf{H}$, $d(z_1, z_2)$ has

- to obey metric axioms: positive, triangle,.....
- has to be $PSL(2, \mathbf{R})$ + conjug.-nvariant, i.e. invariant with respect to transformations (2.6)
- be additive on geodesics

Theorem 6. *Let G be a Riemannian metric which is invariant with resepct to the action of the group* (2.6) *of* $\mathbf{H}$. *Then up to multiplier is defined by equation*

$$G = \frac{dz d\bar{z}}{(z - \bar{z})(\bar{z} - z)}$$

This theorem follows from the lemma

Lemma 2. *Let $d = d(ia, ib)$ be metric on vertical line $y = 0$, $(a, b > 0)$. Then*

$$d(ia, ib) = \left| \log \left(\frac{a}{b} \right) \right| ,$$

since using group of isometry we can define the distance between two arbitrary points.

Prove the lemma

Proof. Transformation $z \to \lambda z$, and $z \to -\frac{\lambda}{z}$ are symmetry transfromations, hence for arbitrary points $z = ia, ib$ and for arbitrary real λ

$$d\left(ia, ib\right) = d\left(i\lambda, i\lambda b\right) = d\left(i\frac{\lambda}{a}, i\frac{\lambda}{b}\right) =$$

Moreover since vertical line is geodesic hence for arbitrary three points ia, ib, ic

$$d(ia, ic) = d(ia, ib) + d(ib, ic)\,, \qquad \text{if } a < b < c$$

It follows from these equations that $d(ia, ib) = \log \frac{a}{b}$

□