Chebyshev approximation and Helly's Theorem .

Hovhannes Khudaverdian

University of Manchester, Manchester, UK

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Helly's Theorem

Chebyshev approximation

- Distance between functions. Minimax polynomials
- Minimax polynomials for polynomial functions.— Chebyshev polyn
- Minimax polynomials for arbitrary functions

Helly's Theorem. Formulation

Theorem

Suppose that there is a family of N convex sets in affine space $\mathbf{R}^n$ such that any $n+1$ sets of this family have a common point. Then these all sets have a common point. (We suppose that $N \geq n+1$).

Helly's Theorem. Special case $n = 1$

Convex set $=$ Interval

Theorem

A finite family of intervals on the line have a common point if arbitrary two intervals of this family have a common point.

Proof. Let $\{[a_i, b_i]\}$ be such intervals. Consider indices i_0, k_0 such that $a_{i_0} = \max_i \{a_i\}$ and $b_{k_0} = \min_k \{b_k\}$.
All left ends of intervals, a_i are less or equal to a_{i_0} and all right ends of intervals b_k are greater or equal to b_{k_0}.
If $a_{i_0} > b_{k_0}$ then intervals $[a_{i_0}, b_{i_0}]$ and $[a_{k_0}, b_{k_0}]$ do not intersect. Hence $a_{i_0} \leq b_{k_0}$. In this case all the points between a_{i_0} and b_{k_0} belong to all intervals ■

Helly's Theorem. Special case $n = 2$

Theorem

Suppose that there is a family of $N \geq 3$ convex sets in the plane such that any 3 sets of this family have a common point. Then these all sets have a common point.

Sketch of the proof for $d = 2$

Let A_1, A_2, A_3, A_4 are four *convex* sets such that any three of them have non-empty intersection. Consider points M_1, M_2, M_3, M_4 such that

$$\begin{cases} M_4 \in A_1 \cap A_2 \cap A_3 \\ M_3 \in A_1 \cap A_2 \cap A_4 \\ M_2 \in A_1 \cap A_3 \cap A_4 \\ M_1 \in A_2 \cap A_3 \cap A_4 \end{cases}$$

Considering configuration of these points one can easy find a point M_0 which belongs to all sets.

Counterexamples

Both conditions

all sets are *convex*

Sets are in $\mathbf{R}^n$

are highly important. One can easy construct counterexamples

Helly's Theorem and Chech cohomology

A family $\{U_1, \ldots .U_N\}$ of sets such that arbitrary three intersect and their interection is empty define 2-cocycle $c_{\alpha\beta\gamma}$.

If $U_1 \cap U_2 \cap U_3 \cap U_4 = \emptyset$ then these sets define 2-cocycle which implies that the cohomology group H^2 is not trivial

Biography

Edward Helly was born in Vienna on June 1, 1884. He awarded PhD in 1907. Before Grand War he published few but very important papers. In particular in 1912 he proved the seminal result which now days may be called as the special case of Hahn-Banach Theorem.

So called Helly's Theorem on convex bodies was discovered by him in 1913. Then war begins...

Helly joined the army. He was mortally wounded. Miraculously survived and captured by Russians, Helly spent 4 years as war prisoner in Russian hospitals and camps. In 1918 Grand war was finished, but another war began in Russia. It was civil war. Famine, total chaos... He reached Vladivostok After two long years Helly returned to Vienna through Asia.

Biography

He recommenced his research and obtained new very strong results. Unfortunately he could not find University work. He began to work in actuarial field and continued his mathematical research.
In 1938 after Anschluss (annexation of Austria in Nazi Germany)
Helly, he was Jewish , was forced to emigrate to America. His life here was not easy too. Only due the help of Albert Einstein he managed to find University job. Helly died in Chicago in 1943.

Helly's theorem is foundation of convex geometry.
We just quote few important results which follow from Helly Theorem.

Blumental, Valin, M. Borgmann

Set of points $\{A_1, A_2, A_3, \ldots, A_n\}$ on the plane belongs to a circle of radius 1 if every three points $\{A_{i_1}, A_{i_2}, A_{i_3}\}$ belong to a circle of radius 1.

Minkovski-Radon Theorem

Let D be a convex bounded set in plane. Then there exists a point $M \in D$ such that for every chord AB which passes through the point M,

$$\frac{AM}{MB} \leq 2.$$

Krasnoselskij Theorem

Theorem

Let Z be a domain in the plane such that for every three points $M_1, M_2, M_3 \in Z$ there exists a point P such that intervals PM_1, PM_2, PM_3 belong to Z. Then Z is star set.

We say that the set Z is *star* set if there exists a point $O \in Z$ such that for every point $M \in Z$ the interval $[OM]$ belongs to the domain Z. (Convex bodies are evidently star sets).

Parallel segments...

Consider a finite set of parallel intervals in the plane, such that for every three intervals there exists a line which intersects these three intervals. Then there exists a line which intersects all the intervals.

Distance between functions

Consider space $C([a,b])$ of continuous functions on the closed interval $[a,b]$. Distance between functions:

$$d(f,g) = ||f-g||_\infty = \max_{x\in[a,b]} |f(x)-g(x)|.$$

Linear spaces of polynomials

Consider the linear space V_n of polynomials of order at most n.

$$V_n = \{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, a_0, \ldots, a_n \in \mathbf{R}\}.$$

Dimension of V_n is equal to $n+1$
V_1 is 2-dimensional linear space of functions $y = px + q$ ('lines')
V_2 is 3-dimensional linear space of functions $y = ax^2 + px + q$.
V_3 is 4-dimensional linear space of functions
$y = dx^3 + ax^2 + px + q$.
and so on....

Affine spaces of polynomials

Sometimes we consider affine space A_n of polynomials of order n with leading term x^n.

$$A_n = \{x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n, a_0, \ldots, a_n \in \mathbf{R}\}.$$

Dimension of A_n is equal to n
A_1 is 1-dimensional affine space of functions $y = x + q$
A_2 is 2-dimensional affine space of functions $y = x^2 + px + q$.
A_3 is 3-dimensional affine space of functions
$y = x^3 + ax^2 + px + q$.
and so on

Approximation by polynomials. Minimax polynomials

Let f be continuous function on the closed interval $[a,b]$. $P_f = P_f(x)$ is *minimax polynomial* of order n, if it is polynomial of order at most n, i.e. $P_f \in V_n$ and it is the closest polynomial to the function in the linear space V_n:

$$P_f^{(n)}: \quad \forall P \in V_n, \quad d(f, P_f^{(n)}) \leq d(f, P)$$

In other words $d(f, P_f^{(n)}) = \max_{x\in[a,b]} |P_f^{(n)}(x) - f(x)|$ attains the minimum value on this polynomial, compared with all polynomials of order at most n.

$$d(f, P_f^n) = \min_{a_0,a_1,\ldots,a_n} \max_{x\in[a,b]} |(a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n) - f(x)|.$$

Minimax poynomial of order 1, $P_f^{(1)}$—closest line L_f.

$$d(L_f, f) = \min_{p,q} \max_{x \in [a,b]} |f(x) - px - q|.$$

$$\forall L, \qquad d(L_f, f) \le d(L, f).$$

Example $f = x^2$ on $[-1, 1]$

Approximation by 'lines' (polynomials of order ≤ 1) The closest line is $L_f = P_f^{(1)} = \frac{1}{2}$,

$$d(f, L_f) = \max_{-1 \leq x \leq 1} \left| x^2 - \frac{1}{2} \right| = \frac{1}{2}.$$

For an arbitrary line $L: \; y = px + q$

$$d(f, L) = \max_{-1 \leq x \leq 1} |x^2 - px - q| \geq \frac{1}{2}.$$

Example $f = x^2$ on $[0, 1]$

In this case the closest line is $L_f = P_f^{(1)} = x - \frac{1}{8}$:

$$d(f, L_f) = \max_{0 \le x \le 1} \left| x^2 - x + \frac{1}{8} \right| = \frac{1}{8}.$$

For an arbitrary line $L: \; y = px + q$

$$d(f, L) = \max_{0 \le x \le 1} |x^2 - px - q| \ge \frac{1}{8}.$$

Example $f = x^3$ on $[-1, 1]$

Approximation by 'parabolas' (polynomials of order ≤ 2) The closest parabola is $P_f^{(2)} = \frac{3}{4}x$,

$$d(f, P_f^{(2)}) = \max_{-1 \leq x \leq 1} \left| x^3 - \frac{3}{4}x \right| = \frac{1}{4}.$$

For an arbitrary parabola $y = ax^2 + bx + c$

$$\max_{0 \leq x \leq 1} |x^3 - ax^2 - bx - c| \geq \frac{1}{4}.$$

Example $f = x^3$ on $[0, 1]$

Approximation by 'parabolas' (polynomials of order ≤ 2) The closest parabola is $P_f^{(2)} = \frac{3}{2}x^2 - \frac{9}{16}x + \frac{1}{32}$,

$$d(f, P_f^{(2)}) = \max_{-1 \leq x \leq 1} \left| x^3 - \left(\frac{3}{2}x^2 - \frac{9}{16}x + \frac{1}{32} \right) \right| = \frac{1}{32}.$$

For an arbitrary parabola $y = ax^2 + bx + c$

$$\max_{0 \leq x \leq 1} |x^3 - ax^2 - bx - c| \geq \frac{1}{32}.$$

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For an arbitrary line $L: \; y = px + q$

$$d(f, L) = \max_{-1 \leq x \leq 1} |x^2 - L(x)| \geq \frac{1}{2}.$$

$$\Delta = f(x) - L_f(x) = x^2 - \frac{1}{2} = \frac{1}{2} \cos 2 \arccos x .$$

Difference Δ attains maximum and minumum values $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

Example $f = x^2$ on $[0,1]$

Approximation by 'lines' (polynomials of order ≤ 1) The closest line is $L_f = P_f^{(1)} = x - \frac{1}{8}$:

$$d(f, L_f) = \max_{0 \leq x \leq 1} \left| x^2 - x + \frac{1}{8} \right| = \frac{1}{8}.$$

For an arbitrary line $L: \ y = px + q$

$$d(f, L) = \max_{0 \leq x \leq 1} |x^2 - L(x)| \geq \frac{1}{8}.$$

$$\Delta = f(x) - L_f(x) = x^2 - x + \frac{1}{8} = \frac{1}{8} \cos 2 \arccos(2x - 1).$$

Difference Δ attains maximum and minumum values $\left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}\right)$.

Example $f = x^3$ on $[-1,1]$

Approximation by 'parabolas' (polynomials of order ≤ 2) The closest parabola is $P_f^{(2)} = \frac{3}{4}x$,

$$d(f, P_f^{(2)}) = \max_{-1\leq x\leq 1} \left| x^3 - \frac{3}{4}x \right| = \frac{1}{4}.$$

For an arbitrary parabola $y = ax^2 + bx + c$

$$\max_{0\leq x\leq 1} |x^3 - ax^2 - bx - c| \geq \frac{1}{4}.$$

$$\Delta = x^3 - \frac{3}{4} = \frac{1}{4}\cos 3\arccos x.$$

Δ attains maximum and minumum values $\left(-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right)$

Example $f = x^3$ on $[0,1]$

Approximation by 'parabolas' (polynomials of order ≤ 2) The closest parabola is $P_f^{(2)} = \frac{3}{2}x^2 - \frac{9}{16}x + \frac{1}{32}$,

$$d(f, P_f^{(2)}) = \max_{-1 \leq x \leq 1} \left| x^3 - \left(\frac{3}{2}x^2 - \frac{9}{16}x + \frac{1}{32} \right) \right| = \frac{1}{32}.$$

For an arbitrary parabola $y = ax^2 + bx + c$

$$\max_{0 \leq x \leq 1} |x^3 - ax^2 - bx - c| \geq \frac{1}{32}.$$

$$\Delta = \left(x^3 - \left(\frac{3}{2}x^2 - \frac{9}{16}x + \frac{1}{32} \right) \right) = \frac{1}{32} \cos 3 \arccos(2x - 1).$$

Δ attains maximum and minumum values $\left(-\frac{1}{32}, \frac{1}{32}, -\frac{1}{32}, \frac{1}{32}\right)$

Chebyshev polynomials

$$T_n(x) = \cos n \arccos x, x \in [-1, 1].$$

$$T_1 = x, T_2 = 2x^2 - 1, T_3 = 4x^3 - 3x, \ldots$$

The distance between polynomial $P_n = \frac{T_n}{2^{n-1}}$ and 0 is equal to $\frac{1}{2^{n-1}}$. This polynomial is closest to 0 in the affine space of n-th order polynomials with leading term x^n:

$$\forall P : P = x^n + \ldots ||P||_\infty = \max_{x \in [-1,1]} |P(x)| \geq \max_{x \in [-1,1]} \left| \frac{T_n(x)}{2^{n-1}} \right| = \frac{1}{2^{n-1}}$$

$$T_n(x_k) = (-1)^k \frac{1}{2^{n-1}}, \quad x_k = \cos \frac{2\pi k}{n} k = 0, 1, 2, \ldots, n.$$

It takes alternating maximum and minumum values $n+1$ times.

Example $T_3(x)$

Polynomial $T_3(x) = \cos 3 \arccos x = 4x^3 - 3x$
$(\cos\varphi = 4\cos^3\varphi - 3\cos\varphi)$

$$\text{Roots:} \quad \left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$$

It attains its maximum and minimum values 4 times at points

$$\begin{array}{cccc} x_0 = -1, & x_1 = -\frac{1}{2}, & x_2 = \frac{1}{2}, & x_3 = 1 \\ T_3(x_0) = -1, & T_3(x_1) = 1, & T_3(x_2) = -1, & T_3(x_3) = 1 \end{array}$$

Polynomial $P_3 = \frac{T_3}{4} = x^3 - \frac{3}{4}x$ is on the smallest distance from 0 in the affine space of all cubic polynomials with leading term x^3.

Chebyshev equi-oscillation Theorem

Let f be an arbitrary continuous function on the interval $[a, b]$.

Theorem

Minimax polynomial $P_f^{(n)}$ is uniquely defined by the condition that there exist $n + 2$ points in which the difference $f(x) - P_f^{(n)}(x)$ attains maximum values with alternating signs.

Necesssry condition can be proved using Helly's Theorem.

Chebyshev equi-oscillation Theorem (n=1)

Let $n = 1$.
Let a line L_f be a closest line to the function f. (L_f is minimax polynomial $P_f^{(1)}$ for $n = 1$).
Then there exist 3 points $x_1 < x_2 < x_3 \in [a, b]$ such that

$$\begin{cases} f(x_1) - L_f(x_1) = \varepsilon \\ f(x_2) - L_f(x_2) = -\varepsilon \\ f(x_3) - L_f(x_3) = \varepsilon \end{cases}, \quad \text{or} \quad \begin{cases} f(x_1) - L_f(x_1) = -\varepsilon \\ f(x_2) - L_f(x_2) = \varepsilon \\ f(x_3) - L_f(x_3) = -\varepsilon \end{cases},$$

where ε is a distance between function f and the line L_f.
We show it using Helly's Theorem.

Revenons à nos moutons: Helly's Theorem again

Consider again the following Corollary of Helly's Theorem:

Consider finite set of parallel intervals on the plane, such that for every three intervals there exists a line which intersects these three intervals. Then there exists a line which intersects all the intervals.

Due to Helly Theorem this result follows from the fact that

A set of lines which intersects given interval can be naturally considered as a convex set in the plane

Let L_f be closest line to the function f. Let $d(f, L_f) = \varepsilon$. Consider a family $\mathscr{M}_\varepsilon$ of vertical intervals centred at the points of graph of the function f. For arbitrary line L, $d(f, L) \geq \varepsilon$, Using continuity arguments and the Corollary of Helly's Theorem we come to observation:

There exist three points x_1, x_2, x_3 such that for an arbitrary line L,

$$|L(x_1) - f(x_1)| \geq \varepsilon \text{ or } |L(x_2) - f(x_2)| \geq \varepsilon \text{ or } |L(x_3) - f(x_3)| \geq \varepsilon .$$

Apply this statement to the closest curve L_f. We come to

$$\begin{cases} f(x_1) - L_f(x_1) = \varepsilon \\ f(x_2) - L_f(x_2) = -\varepsilon \\ f(x_3) - L_f(x_3) = \varepsilon \end{cases}, \quad \text{or} \quad \begin{cases} f(x_1) - L_f(x_1) = -\varepsilon \\ f(x_2) - L_f(x_2) = \varepsilon \\ f(x_3) - L_f(x_3) = -\varepsilon \end{cases}.$$

Example: Closest line L_f for $f = \sin x$ on $[0, \pi/2]$

$L_f: \; y = kx + b$, $d(L_f, \sin x) = \varepsilon$,

$$\varepsilon = (kx + b - \sin x)\big|_{x_1=0} = -(kx + b - \sin x)\big|_{x_2} = (kx + b - \sin x)\big|_{x_3=\pi/2},$$

$b = \varepsilon, k\frac{\pi}{2} + b - 1 = \varepsilon$, the middle point x_2 is defined by the stationary point condition: $\sin x_0 - kx_0 - b = \varepsilon, \quad \cos x_0 = k$. Solving these equations we come to $k = \frac{2}{\pi}$, $x_0 = \arccos\frac{2}{\pi}$ and

$$\varepsilon = b = \frac{1}{2}\left(\sin\arccos\frac{2}{\pi} - \frac{2}{\pi}\arccos\frac{2}{\pi}\right) = \frac{\sqrt{\pi^2 - 4} - 2\arccos\frac{2}{\pi}}{2\pi} \approx 0.105$$

The line $y = \frac{2}{\pi}x + b$ with $b \approx 0.10526$ is the closest line to the function $f = \sin x$ on the interval $[0, \pi/2]$. The distance is equal to $\varepsilon \approx 0.10526$.

References

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2. V.Protasov. Helly's Theorem and about... (in Russian) (see webpage www.geometry.ru/articles/protasov_pdf)

3. Chebyshev approximation and Helly's Theorem (see my webpage www.maths.mathematics/khudian/Etudes/Algebra/chebyshev2.pdf)