On one theorem on quadratic forms and free particles.

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`Galois Lecture`

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The talk is devoted to the following fact: Every mechanical system in a vicinity of stability point can be described as a system of non-interacting oscillators. To explain it we consider a system of interacting N particles, Every particle has three degrees of freedom (three coordinates). This system can be described by second order Newton differential equations on $3N$ coordinates. Suppose that this system is 'almost' in equilibrium position. Using the linear algebra one can introduce new $3N$ coordinates $y^1, \dots, y^{3N}$ so called collective coordinates in this $3N$-dimensional dynamical system, such that every coordinate y^i will obey the differential equation,

$$\frac{d^2 y^i(t)}{dt^2} + \omega^2 y^i = 0\,. \tag{0.1}$$

This is the equation for harmonic oscillator. Thus we see that in these new coordinates the system is described by $3N$ free oscilators, the osillators, which do not interact with each other. These free harmonic osillators, may be interpreted as non-relativistic version ('phonons') of photons[1)]*.*

`Physiscts consider Harmonic osillator as a fundamental physical system. Why? Because every system in linear approximmation behaves as a collection of free oscillators.`

Consider an arbitrary system of interacting N particles $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$. Every particle has three degrees of freedom (three coordinates). This system can be described by second order Newton differential equations on $3N$ coordinates:

$$m_i \frac{d^2 \mathbf{r}_i(t)}{dt^2} = \mathbf{F}_i. \tag{1.1}$$

(for every $i = 1, \dots, N$ we have three second order equations.)

Suppose that all forces are potential, i.e. there exist a function $U = U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n)$ such that the force acting on i-th particle is equal to

$$\mathbf{F}_i = -\frac{\partial U(\mathbf{r}_1, \dots, \mathbf{r}_n)}{\partial \mathbf{r}_i} \tag{1.2}$$

A function $U = U(\mathbf{r}_1, \dots, \mathbf{r}_N)$ is called *potential energy* function.

1) *The material of the lecture will appear in my personal homepage "khudian.net" in the subdirectory Lectures/Galois group lectures*

Suppose that the system is 'almost' in equilibrium position. What it means? Consider a set of vectors, $\{\mathbf{r}_1^{(0)}, \mathbf{r}_2^{(0)}, \dots \mathbf{r}_{N-1}^{(0)}, \mathbf{r}_N^{(0)}, \}$, such that if i-th particle is at the point $\mathbf{r}_i^{(0)}$ for all $i = 1, 2, \dots, N-1, N$ then all the forces acting on every particle vanish, i.e. this becomes an *equilibrium configuration*: all the particles are in the equilibrium position. This means that the configuration $\{\mathbf{r}_1^{(0)}, \mathbf{r}_2^{(0)}, \dots \mathbf{r}_{N-1}^{(0)}, \mathbf{r}_N^{(0)}\}$ is a stationary point of potential energy function:

$$\mathbf{F}_i = -\frac{\partial U}{\partial \mathbf{r}_i}\Big|_{\mathbf{r}_1=\mathbf{r}_1^{(0)}, \mathbf{r}_2=\mathbf{r}_2^{(0)}, \dots, \mathbf{r}_{N-1}=\mathbf{r}_{N-1}^{(0)}, \mathbf{r}_N=\mathbf{r}_N^{(0)}} = 0\,. \tag{1.3}$$

every particle $\mathbf{r}_i(t)$ oscillates around the stability point $\mathbf{r}_{(0)}$ [2)].

Our aim is to study the solutions $\mathbf{r}_i(t)$ of equation (1) in the case of every particle $\mathbf{r}_i(t)$ is almost in equilibrium position (3).

('Almost' means that oscillations are small.)

It is useful to rewrite equations (1.1) as equations for Lagrangian:

$$m_i \frac{d^2\mathbf{r}_i(t)}{dt^2} = \mathbf{F}_i = -\frac{\partial U(\mathbf{r}_1, \dots, \mathbf{r}_n)}{\partial \mathbf{r}_i} \Leftrightarrow \frac{\partial L(\mathbf{r}_i, \mathbf{v}_j)}{\partial \mathbf{r}_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}_i}\right) \tag{1.4}$$

where

$$L = L(\mathbf{r}_i, \mathbf{v}_i) = \text{Kinetic energy} - \text{Potential energy} = \sum_{k=1}^{N} \frac{m_k \mathbf{v}_k^2}{2} - U(\mathbf{r}_1, \dots, \mathbf{r}_N)\,. \tag{1.4a}$$

Equations (1.1) are invariant only with respect to orthogonal trasnformations. Euler Larange equations (1.1L) on Lagrangian (5) accept arbitrary transformations of coordinates..

Change little bit notations

Following traditions denote the set of all coordinates by q^i (where $i = 1, \dots, 3N$)

$$\left(\underbrace{\mathbf{r}_1}_{\text{first particle}}, \underbrace{\mathbf{r}_2}_{\text{second particle}}, \dots \underbrace{\mathbf{r}_N}_{N\text{-th particle}}\right) =$$

$$= \left(\underbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}_{\text{first particle}}, \underbrace{\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}}_{\text{second particle}}, \dots \underbrace{\begin{pmatrix} x_N \\ y_N \\ z_N \end{pmatrix}}_{N\text{-th particle}}\right) =$$

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2) is it stable, or instable equilibrium point? This depends on the Hessian matrix $\frac{\partial^2 U}{\partial \mathbf{r}_i \partial \mathbf{r}_k}$ in stationary point (see further.)

$$\left(\underbrace{q^1,q^2,q^3}_{\text{first particle}}\ ,\ \underbrace{q^4,q^5,q^6,}_{\text{second particle}}\ ,\dots \underbrace{q^{3N-2},q^{3N-1},q^{3N}}_{N\text{-th particle}}\right), \tag{1.5}$$

Respectively we will rewrite Lagrangian and motion equations

$$L(\mathbf{r}_1,\dots \mathbf{r}_N,\mathbf{v}_1,\dots,\mathbf{v}_N) \longrightarrow L(q^1,\dots q^{3N},\dot q_1,\dots q_{3N}) = \sum_{k=1}^{3N} \frac{m_k \dot q_k^2}{2} - U(q_1,\dots,q_{3N})\,. \tag{1.4c}$$

One can consider the system of $3N$ 'one-dimensional particles' which oscillate around the stability point.

Remark one can see from equation (?) that for every $k = 1,\dots N$ coordinates $(q_{3k}, q_{3k+1}, q_{3k+1})$ are coordinates of the k-th particle, i.e. in equation (7) 'masses' $m_{3k} = m_{3k+1} = m_{3k+2}$.

Remark In geometry usually we use upper indices for coordinates. Beacuse of technical reasons we use lower and upper indices...

Remark A good example of such a system is the system of particles whic are loclaised in the nodes of integer lattice.

For potential energy $U = U(q_1,\dots,q_{3N})$ we have according to (3) that

$$\frac{\partial U(q_1,\dots,q_N)}{\partial q_i}\Big|_{q_i=q_i^{(0)}} = 0\,, \tag{8}$$

where we denote by $q_i^{(0)}$ the stationary value of the coordinate q_i (see considerations in equation (6)); E.g. $\begin{pmatrix} q_1^{(0)} \\ q_2^{(0)} \\ q_3^{(0)} \end{pmatrix} = \mathbf{r}_1^{(0)}$.

Remark If we suppose that these are stable equilibrium points, this means that corresponding hessian is non-degenerate and it is positive defnite.

Consider the Taylor expansion for the function U in a vicinity of stationary point.

$$U(q_1,\dots q_{3N}) = U(q_1^{(0)},\dots q_{3N}^{(0)}) + \sum_{k=1}^{3N} \frac{\partial U(q_1,\dots,q_{3N})}{\partial q^k}\Big|_{q_i=q_i^{(0)}} \cdot (q_k - q_k^{(0)}) +$$

$$+\frac{1}{2}\sum_{k,m=1}^{3N} \frac{\partial^2 U(q_1,\dots,q_{3N})}{\partial q_k \partial q_m}\Big|_{q_i=q_i^{(0)}} \cdot (q_k - q_k^{(0)}) + \cdot (q_m - q_m^{(0)}) + o\left(\left(q-q^{(0)}\right)^2\right) \tag{1.6a}$$

Since we are in stability point then due to (8) this formula can be simplified:

$$U(q_1,\dots q_{3N}) = U(q_1^{(0)},\dots q_{3N}^{(0)}) + \frac{1}{2}\sum_{k,m=1}^{3N} \underbrace{\frac{\partial^2 U(q_1,\dots,q_{3N})}{\partial q_k \partial q_m}\Big|_{q_i=q_i^{(0)}}}_{H_{ik}} (q_k - q_k^{(0)})(q_m - q_m^{(0)}) + o\left(\left(q-q^{(0)}\right)^2\right) \tag{1.6b}$$

Now we introduce new coordinates

$$\{x^i\}\colon x^i = q_i - q_i^{(0)} \tag{1.7}$$

This is just translation, which does not affect differential equations (5). We consider small oscillations, thus we will omit terms of order bigger than 2 in equation (11); we omit also constant term in Lagrangian, and we will come to Lagrangian:

$$L(x^1, x^{3N}, \dot{x}^1, \dot{x}^{3N}) = \sum_{k=1}^{3N} \frac{m_k \dot{x}^k \dot{x}^k}{2} - \frac{1}{2} \sum_{k,m=1}^{3N} H_{km} x^k x^m \tag{1.8}$$

In these coordinates differential equations for coordinates x^k look almost the same as for coordinates q_k:

$$m_k \frac{d^2 x^k}{dt^2} + \sum_{m=1}^{3N} H_{km} x^m = 0\,. \tag{1.8b}$$

(we omit higher order terms) Our aim is to see can we come to new coordinates such that in these coordinates thiese equations look simpler.

It is a time to switch on the linear algebra.

Consider vector space $\mathbf{R}^{3N}$ of $3N$-tuples.

To every point with coordinates $(x^1, \dots, x^{3N})$ correspond a vector $\mathbf{x}$. We suppose that the canonical basis $\{\mathbf{e}_i\}$ is defined in $\mathbf{R}^{3N}$, ($\mathbf{e}_i$, $i = 1, \dots, 3N$ is $3N$-tuple such that the i-th component of this vector is eual to 1, and all other components are equal to zero.

The dynamicla system defines two bilinear symmetric forms in the vector space $\mathbf{R}^{3N}$: First form induced by kinetic energy of the Lagrangian (1.8):

$$T(\mathbf{x}, \mathbf{y})\colon \quad T(\mathbf{x}, \mathbf{x}) = \sum_{k=1}^{3N} \frac{m_k x^k x^k}{2} \tag{2.1a}$$

and the second form induced by the Hessian of potential energy at the equilibrium point:

$$H(\mathbf{x}, \mathbf{y})\colon \quad H(\mathbf{x}, \mathbf{x}) = \sum_{k=1} H_{km} x^k x^m\,, \tag{2,1b}$$

Theorem *Let $T = T(\mathbf{x}, \mathbf{y})$, $H = H(\mathbf{x}, \mathbf{y})$ be two symmetric blinear forms in (finite-dimensional) vector space V, and the first form is positive definite:*

$$T(\mathbf{x}, \mathbf{x}) \geq 0\,, \quad T(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = 0\,. \tag{2.2a}$$

This form defines Euclidean structure in vector space V, i.e. the scalar product:

$$(\mathbf{x}, \mathbf{y}) = T(\mathbf{x}, \mathbf{y})\,. \tag{2.2b}$$

There exist an orthonormal basis $\{\mathbf{f}_i\}$ (with respect to the scalar product (T2)) such that

$$H(\mathbf{e}_i, \mathbf{e}_j) = \{ 0 \text{ if } i \neq j$$

i.e. in this basis matrix of bilinear form T is identity matrix, and matrix of the bilinear form H is the matrix

$$||H|| = \text{diag}\,\{\lambda_1, \lambda_2, \dots, \lambda_n\}\,.$$

Use this Theorem to the Lagrangian. Consider the new coordinates $\{y^i\}$ corresponding to the basis $\{\mathbf{f}_i\}$. We come to Lagrangian

$$L = \sum_{k=1}^{3N} \frac{\dot{y}_k^2}{2} - \sum_{k=1}^{3N} \frac{\lambda_k y_k^2}{2}\,, \tag{2.3}$$

This is the Lagrangian of $3N$ non-interacting particles. Every particle obeys the equation of motion

$$\frac{d^2 y^i(t)}{dt^2} + \omega^2 y^i = 0\,. \tag{2.3b}$$

We come to the equation that we declared above (see equation (0.1)).

If all $\lambda_i > 0$, i.e. (the bilinear form is H positive definite also) then the equilibrium position is stable, and

Now we prove the Theorem

Proof of the Theorem

Formulate

Lemma A symmetric operator P on Euclidean space $\mathbf{E}^n$ has at least one eigenvector. (Operator P is a symmetric operator on $\mathbf{E}^n$ if $(P\mathbf{x}, \mathbf{y}) = (\mathbf{x}, P\mathbf{y})$.

Proof of the Theorem (`based on Lemma`)

Symmetric positive-definite bilinear form $T(\mathbf{x}, \mathbf{y})$ defines in V the Euclidean structure, the scalar product (2.1a). Symmetric bilinear form $H = H(\mathbf{x}, \mathbf{y})$ defines on this Euclidean space a symmetric linear operator P_H such that for arbitrary vectors $\mathbf{x}, \mathbf{y}$ $(P_H(\mathbf{x}), \mathbf{y}) = H(\mathbf{x}, \mathbf{y})$.

Using lemma consider the unit vector $\mathbf{e}_1$ which is proportional to eigenvector of this symmetric operator, and consider the subspace V_1 of vectors orthogonal to the vector $\mathbf{e}_1$: $V_1 = \{\mathbf{x}\colon\ (\mathbf{x}, \mathbf{e}_1) = 0\}$. The subspace V_1 is invariant subspace of operator P_H, and this operator is a symmetric operator on this subspace. We can by induction continue the process on V_1. Thus we will come to the statement of the Theorem.

Remark It is very easy to see that in fact we deduced from the lemma the

Theorem $'$ A symmetric operator on $\mathbf{E}^n$. is diagonalisable, and all its eigenvalues may be chosen to be orthogonal to each other.

It remains prove the lemma.

We will present two proves which look different.

`PROOF OF THE LEMMA`

`First proof`

Let P be a symmetric operator on Euclidean space $\mathbf{E}^n$. Consider the function $f(\mathbf{x}) = (P\mathbf{x}, \mathbf{x})$ on the the unit sphere $S^{n-1} = \{\mathbf{x}\colon, (\mathbf{x}, \mathbf{x}) = 1$.The compactness of the sphere implies the existence of a vector $\mathbf{x}_0$ on the sphere, such that this function atteints the maximum on this vector. This is an eigenvector of operator P.

`Second proof`

Consider complexification $V_{\mathbf{E}^n} = \mathbf{E}^n \times \mathcal{C}$ of initial Euclidean vector space $\mathbf{E}^n$.

Let λ be a root (maybe complex) of polynomial $P(z) = \det(P_H - z)$, and let $\mathbf{x}$ be a corresponding eigenvector (may be complex). Then the condition $(P\mathbf{x}, \mathbf{x}) = \lambda(\mathbf{x}, \mathbf{x}) = (\mathbf{x}, P\mathbf{x}) = \bar{\lambda}(\mathbf{x}, \mathbf{x})$ implies that λ is real, respectively the eigenvector $\mathbf{x}$ maybe chosen also real.

It seems that the second proof is purely algebraic, on the other hand it uses the fundamental theorem of algebra, hence it uses more or less the continuity.