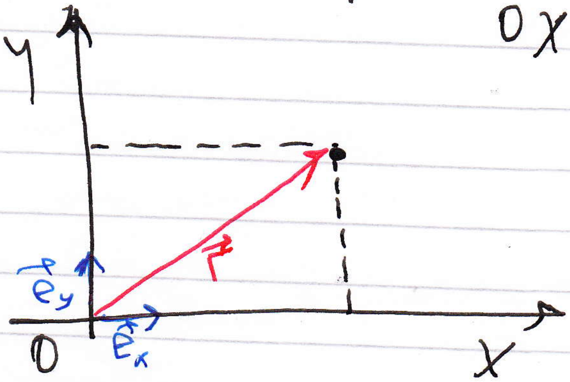
23 March
Lecture CII

Cartesian coordinates
(orthogonal, rectangular)

Transformation from Cartesian coordinates to Cartesian coordinates

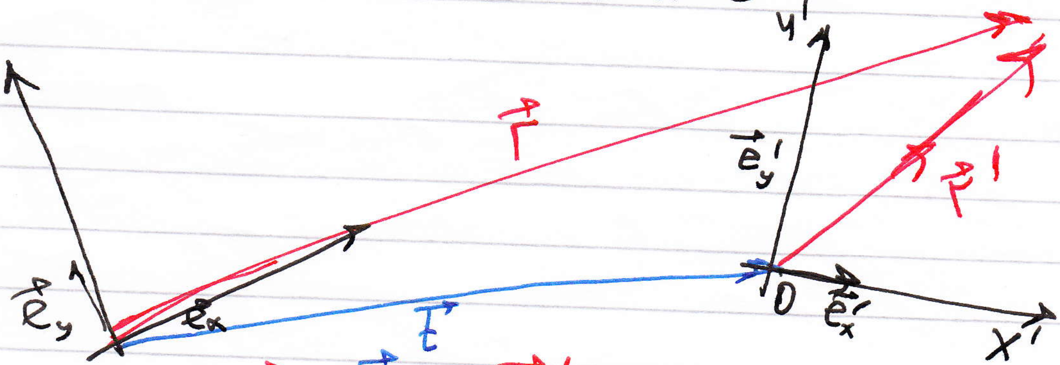
Plane - $\mathbb{E}^2$ Space - $\mathbb{E}^3$

OX, OY - orthogonal lines.



$\vec{r} = x\vec{e}_x + y\vec{e}_y$ $\{\vec{e}_x, \vec{e}_y\}$ - orthonormal basis

(x, y) - Cartesian coordinates



$\vec{r} = \vec{t} + \vec{r}'$

$$(\vec{e}_x, \vec{e}_y)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{r} = \underbrace{(\vec{e}_x, \vec{e}_y)\begin{pmatrix} a \\ b \end{pmatrix}}_{\vec{t}} + \underbrace{(\vec{e}_x', \vec{e}_y')\begin{pmatrix} x' \\ y' \end{pmatrix}}_{\vec{r}'}$$

$$x\vec{e}_x + y\vec{e}_y = \vec{r} = \vec{t} + \vec{r}' = a\vec{e}_x + b\vec{e}_y + x'\vec{e}_x' + y'\vec{e}_y'$$

$$(\vec{e}_x', \vec{e}_y') = (\vec{e}_x, \vec{e}_y)\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$(\vec{e}_x, \vec{e}_y)\begin{pmatrix} x \\ y \end{pmatrix} = (\vec{e}_x, \vec{e}_y)\begin{pmatrix} a \\ b \end{pmatrix} + (\vec{e}_x, \vec{e}_y)\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$\rightarrow$ orthogonal matrix

Transformation from Cartesian Coordinates to another Cartesian coordinates (in plane)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \underbrace{\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}}_{\text{orthogonal}} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{cases} x = a + p_{11}x' + p_{12}y' \\ y = b + p_{21}x' + p_{22}y' \end{cases}$$

Example. $\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

rotation $P^T = P^{-1}$, $\det P = 1$

$$\begin{cases} x = a + x'\cos\varphi - y'\sin\varphi \\ y = b + x'\sin\varphi + y'\cos\varphi \end{cases}$$

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

rotation + reflection $P^T = P^{-1}$, $\det P = -1$

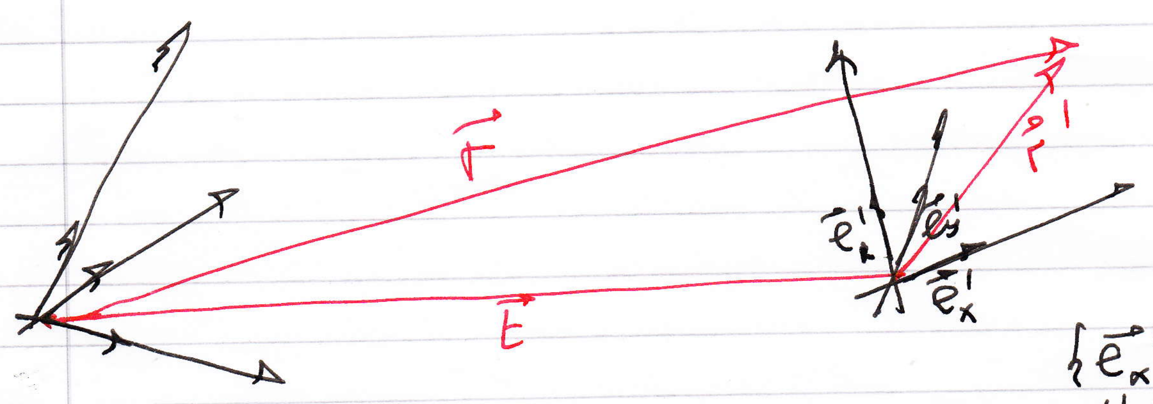
$$\begin{cases} x = a + x'\cos\varphi + y'\sin\varphi \\ y = b + x'\sin\varphi - y'\cos\varphi \end{cases}$$

Cartesian coordinates

Lecture C II

23 March VI

Transformation of coordinates in $\mathbb{E}^3$



$$\vec{r} = \vec{t} + \vec{r}'$$

$$\underbrace{(\vec{e}_x, \vec{e}_y, \vec{e}_z)\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{r}} = \underbrace{(\vec{e}_x, \vec{e}_y, \vec{e}_z)\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\vec{t}} + (\vec{e}_x', \vec{e}_y', \vec{e}_z')\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$(\vec{e}_x', \vec{e}_y', \vec{e}_z') = (\vec{e}_x, \vec{e}_y, \vec{e}_z)\underbrace{\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}}_{\text{orthogonal matrix}}$$

$P^T P = E$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\text{translation}} + \underbrace{\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}}_{\text{rotation + reflection}}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

(or just rotation)

Examples.

Rotation around axis Oz on angle φ

$$\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Rotation around axis OX and translation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{cases} x = a + x' \\ y = b + y'\cos\theta - z'\sin\theta \\ z = c + y'\sin\theta + z'\cos\theta \end{cases}$$

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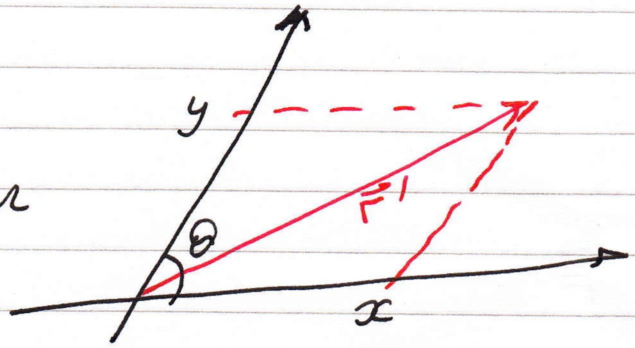
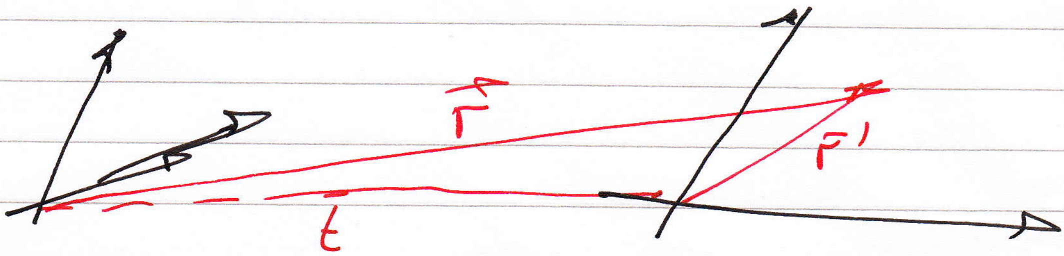
Affine coordinates

Affine coordinates



If angle $\theta = \frac{\pi}{2}$
coordinates are Cartesian



$$\vec{r} = \vec{t} + P\vec{r}\,'$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \boxed{\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Arbitrary invertible matrix

P- is invertible, $\det P \neq 0$

$$\begin{cases} x = a + p_{11}x' + p_{12}y' \\ y = b + p_{21}x' + p_{22}y' \end{cases}$$

If P - orthogonal matrix ($P^T P = E$)
$\det P = \pm 1$

Cartesian —— Cartesian

If P is just (arbitrary) invertible matrix

Affine coordinates —— Affine coordinates