Lecture C III
Analytical definition of conic sections.

We will call ellipse, hyperbola, parabola

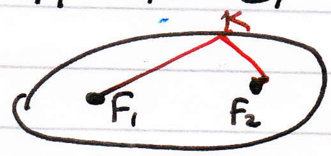
CONIC SECTIONS.

(In the next lecture this terminology will be expleined).

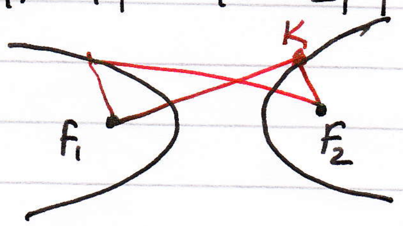
In the first lecture we consider geometrical definition of conic sections.

Recall it:

Ellipse $- \{K: \quad |KF_1| + |KF_2| = 2a\}$



Hyperbola $- \{K: \quad \big||KF_1| - |KF_2|\big| = 2a$



Parabola $- \{K: \ |KF| = d(K, l)\}$

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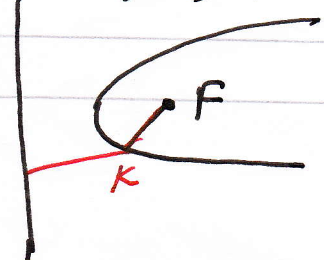


Lecture C III

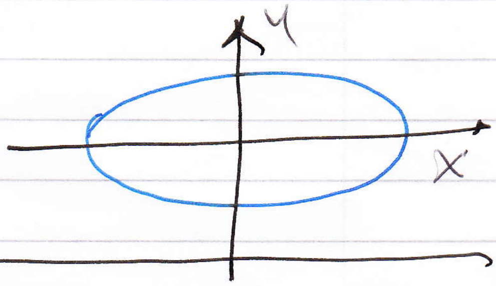
Analytical definition of ellipse, parabola, hyperbola.

Def. Let C be a curve on plane.

A curve C is an ellipse if there exist
Cartesian coordinates (x, y) such that
C is defined by equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

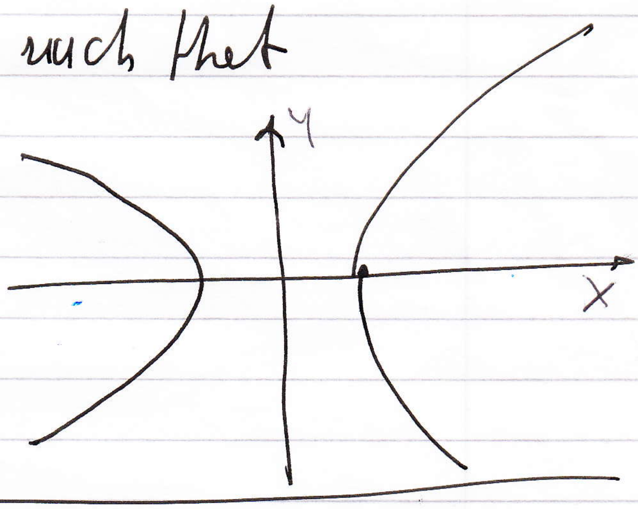
in these Cartesian coordinates



A curve C is an hyperbola if there exist
Cartesian coordinates (x, y) such that
C is defined by equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

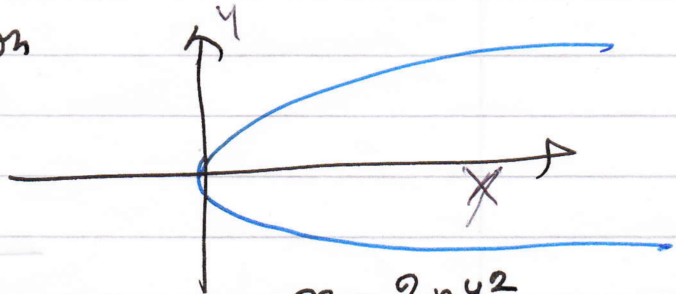
in these Cartesian coordinates



A curve C is parabola if there exist
Cartesian coordinates (x, y) such that
C is defined by equation

$$x = 2py^2$$

in these Cartesian coordinates
~~$(x = py^2)$~~

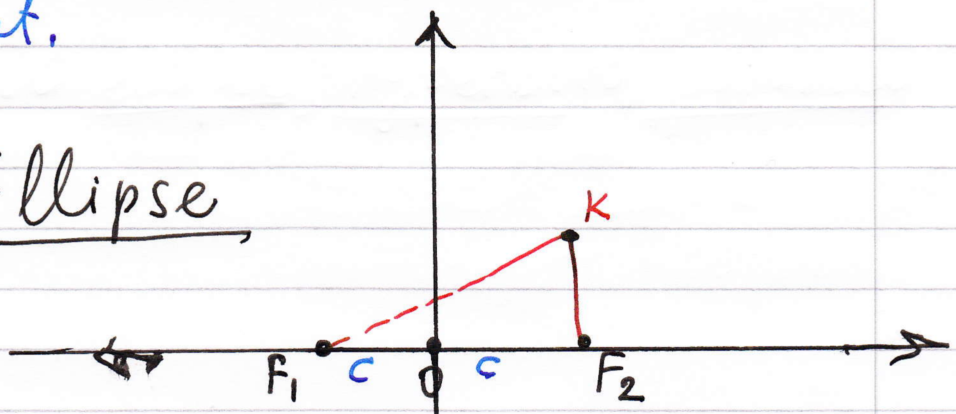


$x = 2py^2$

Lecture C III

Show that geometrical and analytical definitions of conic sections (ellipse, hyperbola, parabola) are equivalent.

Ellipse



Geom. def.

$F_1 = (-c, 0)$; $F_2 = (c, 0)$, $K = (x, y)$

$|KF_1| + |KF_2| = 2a$

We have $|KF_1| = \sqrt{(x+c)^2 + y^2}$
$|KF_2| = \sqrt{(x-c)^2 + y^2}$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \qquad (a \geq c > 0)$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$ Take square:

$$\cancel{x^2} + \underline{2xc} + \cancel{c^2} + \cancel{y^2} = 4a^2 + \cancel{x^2} - \underline{2xc} + \cancel{c^2} + \cancel{y^2} - 4a\sqrt{(x-c)^2 + y^2}$$ Hence

$$\cancel{4}a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4xc = \cancel{4}(a^2 - xc)$$ Take again square:

$$a^2(x^2 - 2xc + c^2 + y^2) = a^4 - 2a^2xc + x^2c^2$$

$$\Downarrow$$

$$\cancel{a^2x^2} \quad (a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2$$

$$\Downarrow$$

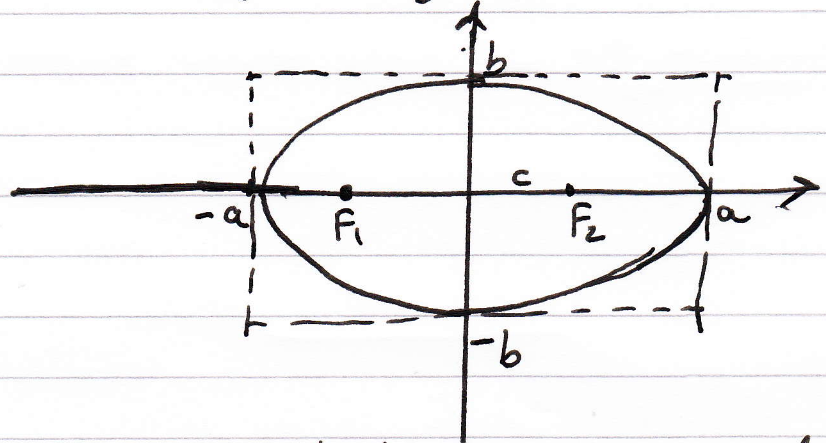
$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

~~We proved, that Geometrical def. ⟹~~

Lecture C III

We proved that all points which belong to locus $\{K: |KF_1| + |KF_2| = 2a\}$ obey equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (b^2 = a^2 - c^2) \quad (*)$$



One can see that converse implication is also true.
Indeed suppose that $(*)$ is obeyed.

Then $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$. We have

$$|KF_1| = \sqrt{(x+c)^2 + y^2} = \sqrt{(x+c)^2 + b^2\left(1 - \frac{x^2}{a^2}\right)} =$$

$$= \sqrt{\underbrace{\left(1 - \frac{b^2}{a^2}\right)}_{c^2/a^2} x^2 + 2cx + \underbrace{c^2 + b^2}_{a^2}} = \sqrt{\left(\frac{c}{a}x + a\right)^2} =$$

$$= \left|\frac{c}{a}x + a\right| = \frac{c}{a}x + a \quad (\text{since } -a < x < a)$$

Analogously $|KF_2| = \sqrt{(x-c)^2 + y^2} = \sqrt{(x-c)^2 + b^2\left(1 - \frac{x^2}{a^2}\right)} =$

$$= \sqrt{\left(1 - \frac{b^2}{a^2}\right)x^2 - 2cx + c^2 + b^2} = \left|\frac{c}{a}x - a\right| = a - \frac{c}{a}x \quad (-a < x < a)$$

Hence

$$|KF_1| + |KF_2| = \left(\frac{c}{a}x + a\right) + \left(a - \frac{c}{a}x\right) = 2a$$

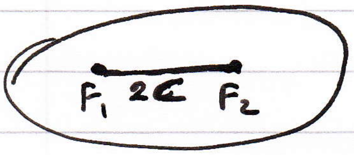
Lecture CIII

We proved both statements.

there exist Cartesian coordinates (x,y):

$$\{K : |KF_1| + |KF_2| = 2a\} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$(b^2 = a^2 - c^2)$$



Thus we see that geometrical and analytical definitions of ellipse are equivalent.

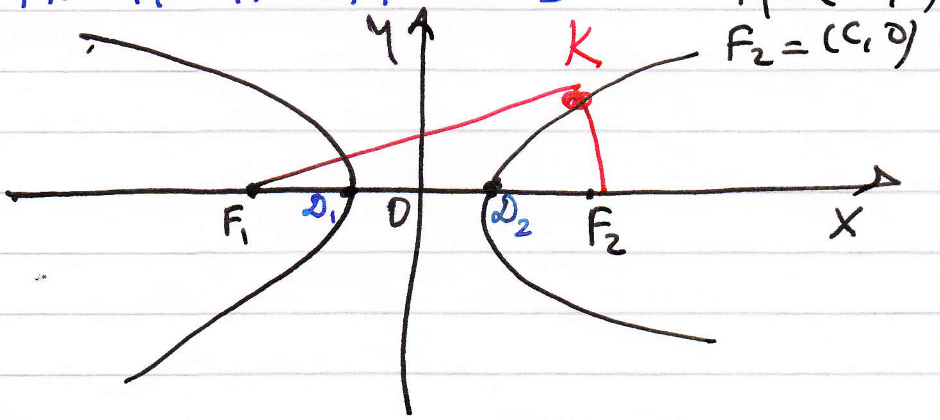
Hyperbola

In the same way one can perform a proof for hyperbola and parabola.

Proof for hyperbola

$$K : \{\,||KF_1| - |KF_2|| = 2a\}$$

$F_1 = (-c, 0)$
$F_2 = (c, 0)$

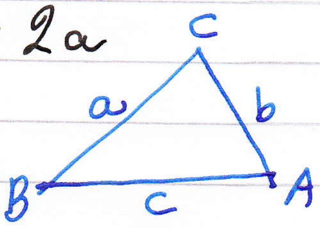


~~We omit the proofs for hyperbola and parabola~~

We choose Cartesian coordinates such that, foci F_1, F_2 are on OX axis, and the origin (point O) is in the middle of the segment $[F_1F_2]$: $|F_1O| = |OF_2| = c$

$$\big||KF_1| - |KF_2|\big| = 2a$$

Note that for any triangle



$a + b > c$
$|a - b| < c$

Hence for hyperbola

$$\big||KF_1| - |KF_2|\big| = 2a < |F_1F_2| = 2c, \quad a < c.$$

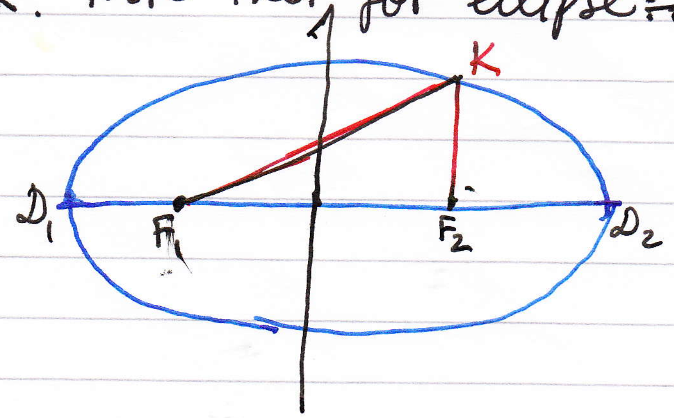
In particular hyperbola intersects OX axis at the points $D_2 = (a, 0)$ and $D_1 = (-a, 0)$

$$\big||D_1F_1| - |D_1F_2|\big| = \big||-a-(-c)| - |-a-c|\big| = |(c-a) - (c+a)| = 2a$$

$$\big||D_2F_1| - |D_2F_2|\big| = \big||a-(-c)| - |a-c|\big| = |(a+c) - (c-a)| = 2a$$

(Foci have coordinates $(-c, 0)$, $(0, c)$

Remark. Note that for ellipse: ~~this~~ it is opposite: $c < a$



$F_1 = (-c, 0)$,
$F_2 = (c, 0)$
$D_1 = (-a, 0)$
$D_2 = (a, 0)$

$$|KF_1| + |KF_2| = 2a > |F_1F_2| = 2c, \quad a > c.$$

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Now: $\big| |KF_1| - |KF_2| \big| = 2a$ (*)

$K = (x, y)$, $F_1 = (-c, 0)$, $F_2 = (c, 0)$

we have

$$\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a \quad \cancel{(*)}$$

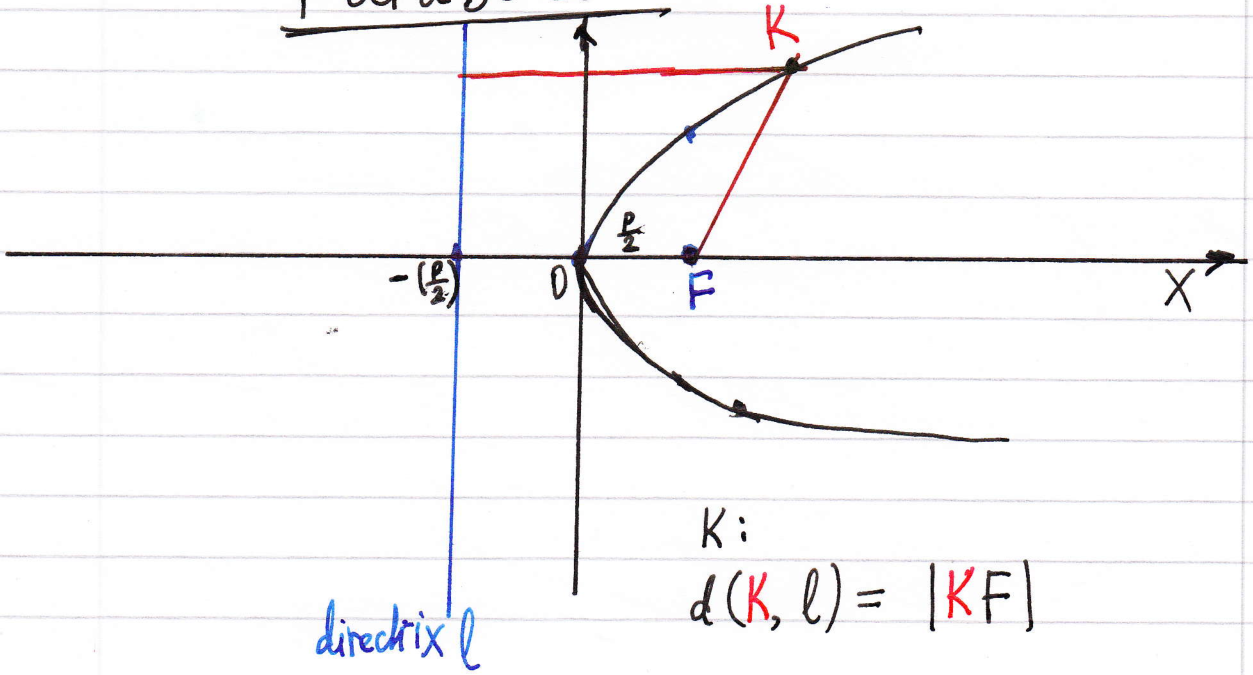
Performing calculations in the same way like for ellipse we will come to the fact that geometrical definition (*) is equivalent to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $\boxed{b^2 = c^2 - a^2}$

i.e. we proved that for hyperbola geometrical and analytical definitions coincide.

Parabola



K:

$d(K, l) = |KF|$

Lecture C $\overline{\underline{III}}$

Choose Cartesian coordinates such that axis OX is passing through focus F and this axis is orthogonal to directrix l , and origin (point O) is between focus and directrix.

$$|OF| = d(O, l) = \frac{p}{2} \quad (d(F, l) = p,\ p > 0).$$

Let $\ \ = (x, y)$ then $d(\ ,\ ) = \left|x + \frac{p}{2}\right|$,

$$|\quad| = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}$$

$$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \left|x + \frac{p}{2}\right|$$

$$\Updownarrow$$

$$\cancel{x^2} - px + \cancel{\frac{p^2}{4}} + y^2 = \cancel{x^2} + px + \cancel{\frac{p^2}{4}}$$

$$\Updownarrow$$

$$\boxed{y^2 = 2px}$$

Thus we show that for Conic Sections (ellipse, hyperbola and parabola) geometrical and analytical definitions are equivalent.