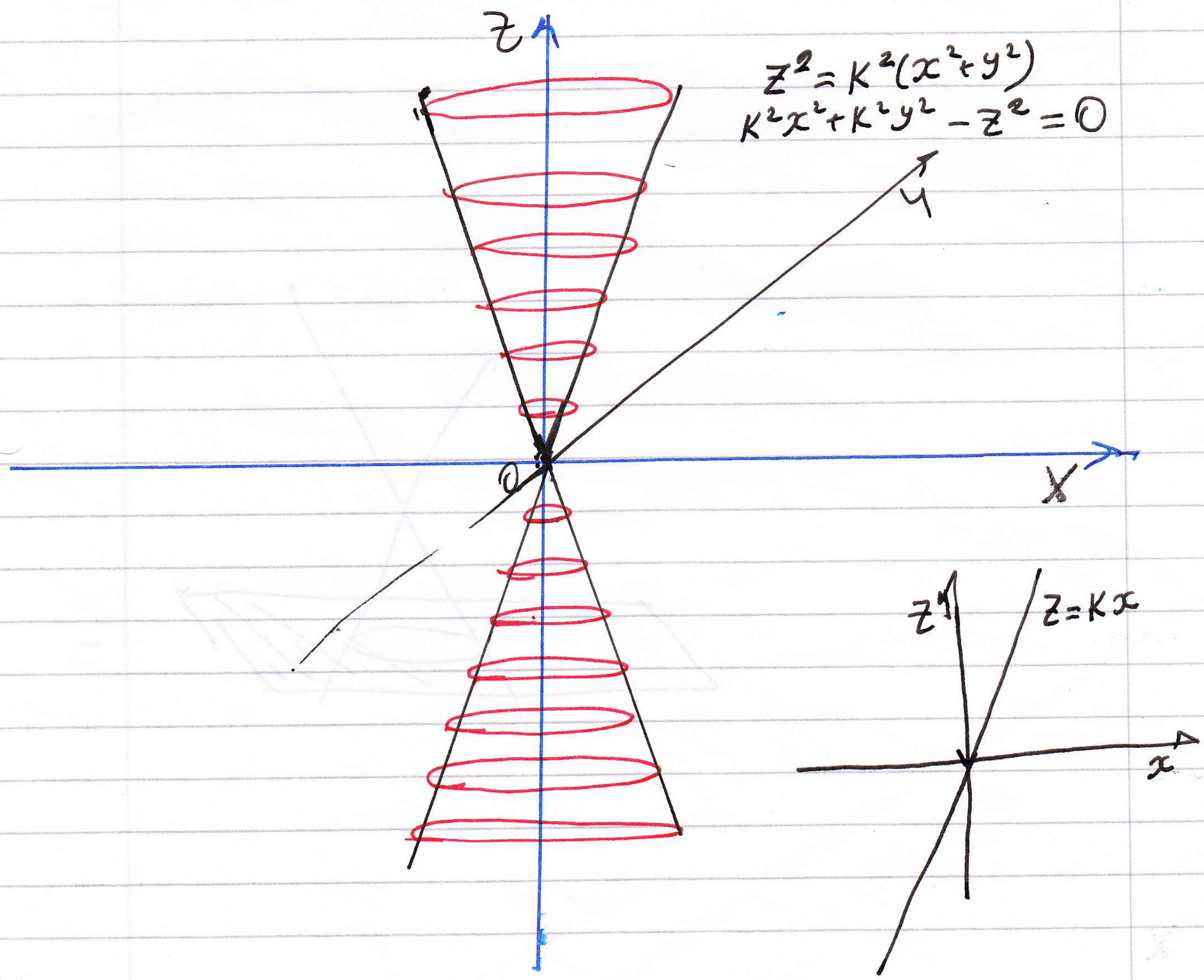
Lecture IV

In this lecture we ~~with~~ consider intersections of planer and surface of cone.

We will show that these intersections are

CONIC SECTIONS

(ie ellipses, or hyperbola or parabola)



Theorem.

Let $\underline{C}$ be a curve which is intersection of a plane with surface of cone.

Let C_{proj} be ~~an~~ orthogonal projection of this curve on the horizontal plane (we suppose that axis of cone is vertical).

Then

a curve C is a conic section[1] (ellipse, or hyperbole or parabole)

a curve C_{proj} is also a conic section[2]

(C_{proj} is ellipse, or hyperbole, or perabole if C is ellipse or hyperbole or perabole, respectively)

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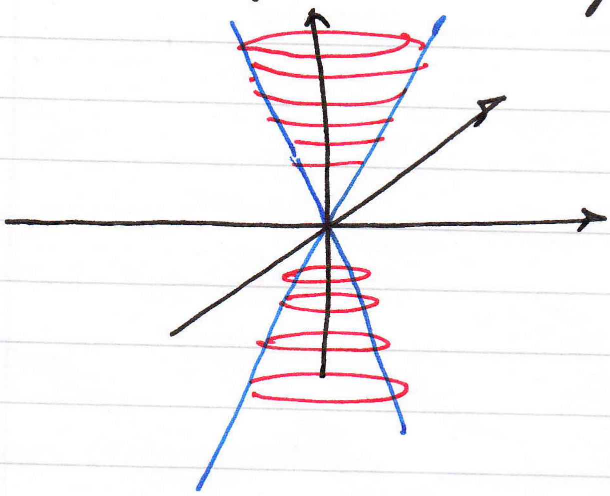
1) We do not consider degenerate cases when C can be just point, all two lines

2) The remarkable property of the curve C_{proj} is that apex (vertex) of the cone is one of foci of this conic section. This implies Kepler law.

Lecture C IV

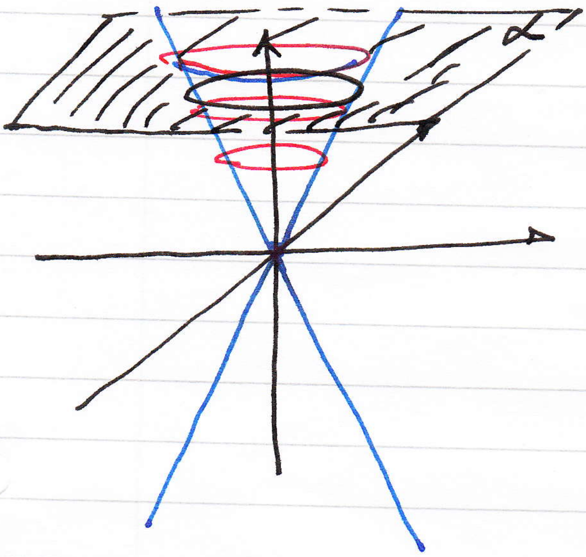
Let conic surface M is given

$$K^2x^2+K^2y^2-Z^2=0.$$



Let α be a plane in $\mathbb{E}^3$

I-st case) plane α is parallel to plane OXY



$$\begin{cases} \alpha: Z=H \\ M: K^2x^2+K^2y^2-Z^2=0 \end{cases}$$

$$\Downarrow$$

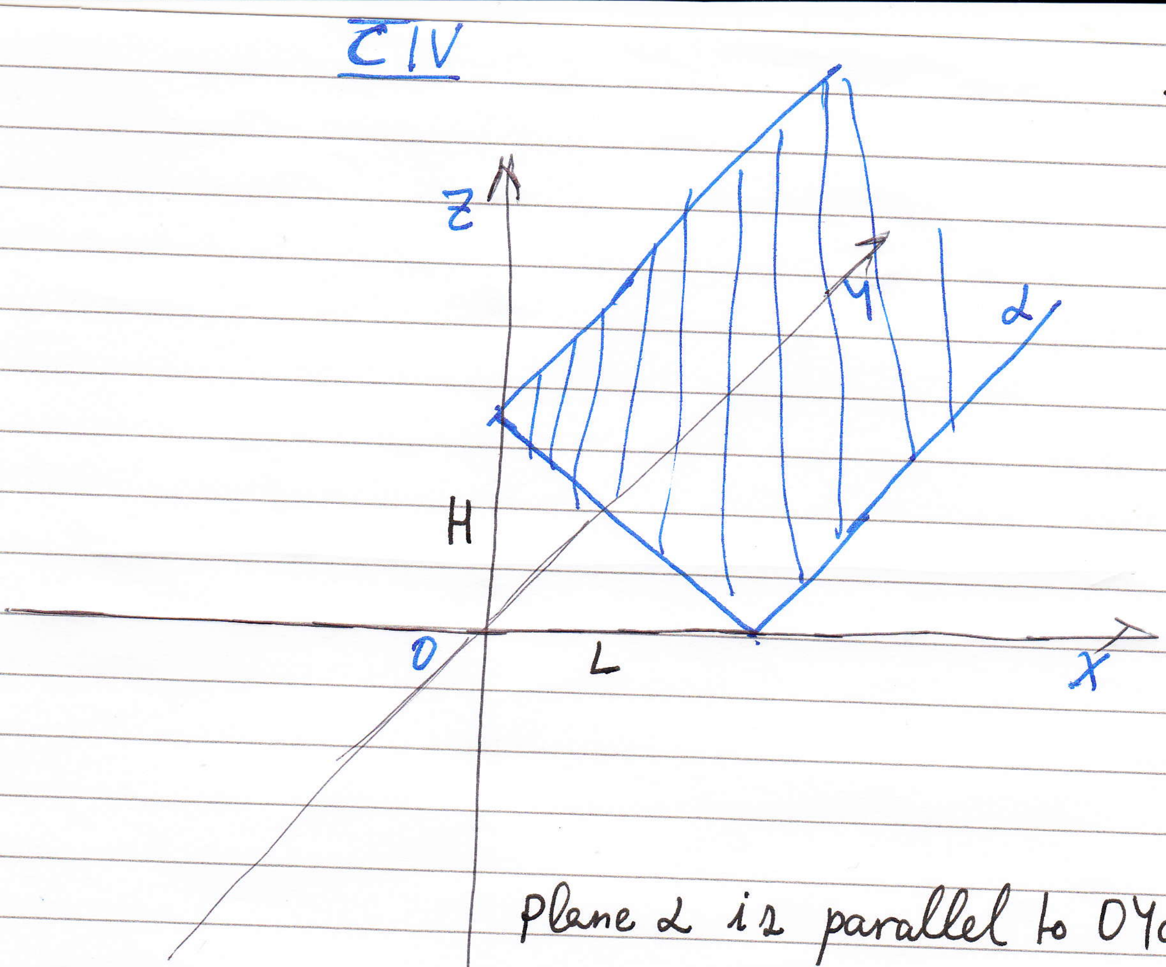
$$\begin{cases} Z=H \\ x^2+y^2=\dfrac{H^2}{K^2} \end{cases}$$

Intersection is the circle of radius $r=\dfrac{H}{K}$

II-nd case) - plane α is not parallel to the plane OXY

In this case ROTATE the space $\mathbb{E}^3$ with respect to axis OZ such that plane α after rotation will be parallel to axis OY

CIV



plane α is parallel to OY axis

It intersects axis OZ at the point $(0,0,H)$ and it intersects axis OX at the point $(L,0,0)$.
(this plane is not parallel to the plane OXY)

Equation of the plane α: $\boxed{\frac{x}{L}+\frac{z}{H}=1}$

(if $x=y=0 \Rightarrow z=H$, if $y=z=0 \Rightarrow x=L$)

Remark. The case when plane α which is not parallel to OXY passes through origin (eg. $ax+bz=0$) is degenerate case. We do not consider it. [In this case plane α intersects with conic by point, apex, or two lines.

Now analyze intersection of plane α with ~~surface of~~ ~~co~~ surface of cone M

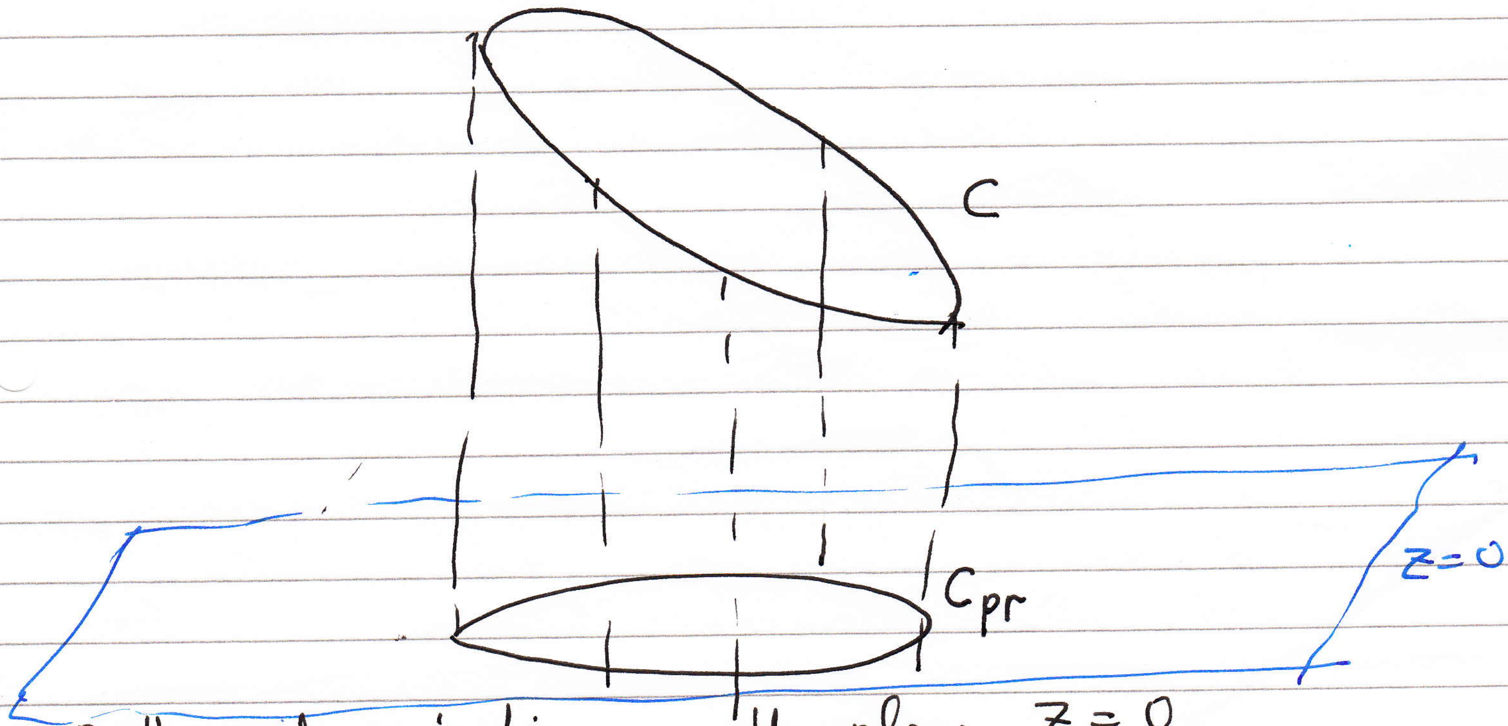
$\alpha \times M$: $\begin{cases} \frac{x}{L} + \frac{z}{H} = 1 \\ K^2x^2 + K^2y^2 - z^2 = 0 \end{cases} \Longleftrightarrow$

intersection of plane α with cone M

$$\Longleftrightarrow \begin{cases} z = H\left(1 - \frac{x}{L}\right) \\ K^2x^2 + K^2y^2 - \left[H\left(1 - \frac{x}{L}\right)\right]^2 = 0 \end{cases}$$

Denote this intersection C.

$$C = \alpha \times M : \begin{cases} z = H\left(1 - \frac{x}{L}\right) \\ K^2x^2 + K^2y^2 - \left[H\left(1 - \frac{x}{L}\right)\right]^2 = 0 \end{cases}$$



Orthogonal projection on the plane $z = 0$

C_{pr}: $K^2x^2 + K^2y^2 - \left(H\left(1 - \frac{x}{L}\right)\right)^2 = 0.$

We first prove that projection, curve C_{pr} is conic section, then we will see that C is conic section also

$$C_{pr}: \quad K^2x^2+K^2y^2-H^2\left(1-\frac{x}{L}\right)^2=0.$$

$$\underbrace{\left(K^2-\frac{H^2}{L^2}\right)}_{\delta=K^2-\frac{H^2}{L^2}}x^2+\frac{2H^2}{L}x+K^2y^2=H^2$$

$$C_{pr}: \quad \delta x^2+\frac{2H^2}{L}x+K^2y^2=H^2.$$

1) $\delta=0$, C_{pr} is parabola: $-\frac{2H^2}{L}x+H^2=K^2y^2$

$$\frac{2H^2}{L}\left(\frac{L}{2}-x\right)=K^2y^2.$$

2) $\delta\neq 0$ in this case

$$C_{pr}: \quad \delta x^2+\frac{2H^2}{L}x+K^2y^2-H^2=\delta\left(x+\frac{H^2}{L\delta}\right)^2+K^2y^2-H^2-\frac{H^4}{L^2\delta}$$

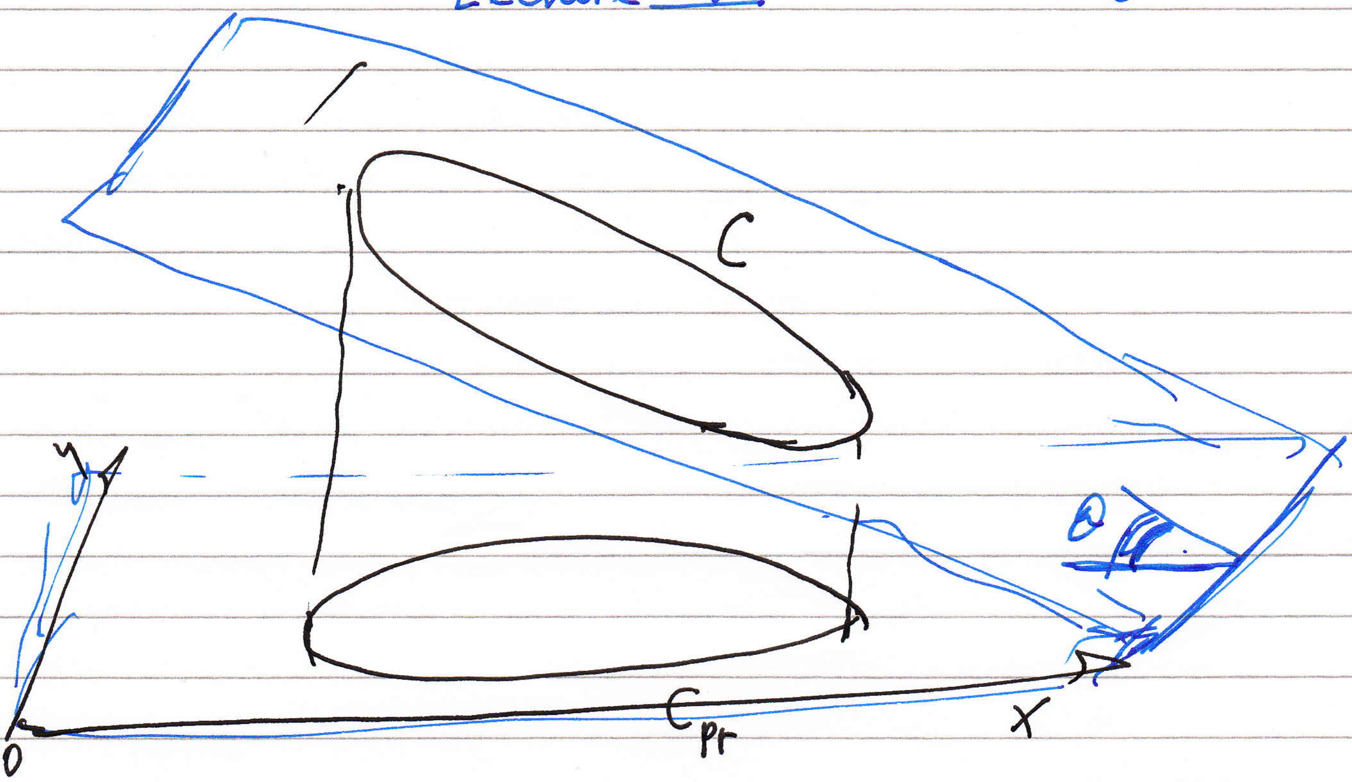
$$C_{pr}: \quad \delta x'^2+Ky^2=H^2+\frac{H^4}{L^2\delta}; \quad \boxed{x'=x+\frac{H^2}{L\delta}}$$

$\delta>0$ — ellipse

$\delta<0$ — hyperbola

Thus we proved that the projection C_{pr} of the curve $C=\alpha\times M$ is conic section.

It remains to prove that curve C is conic section too.



Curve C belongs to plane α: $\frac{x}{L}+\frac{z}{H}=1$

(see the page -3-).

The angle between plane α and plane OXY is $\underline{\theta}$

$$tg\,\theta = \frac{H}{L}$$

Thus if (x,y) are Cartesian coordinates on OXY one can choose Cartesian coordinates $(\tilde{x},\tilde{y})$ on plane α such that

$$\tilde{x} = \frac{x}{\cos\theta}, \quad \tilde{y} = y.$$

Then equation for projection

$$C_{pr}: \ \delta x^2 + \frac{2H^2}{L}x + K^2y^2 = H^2$$

will transform to equation

$$C: \ \delta(\tilde{x}\cos\theta)^2 + \frac{2H^2}{L}(\tilde{x}\cos\theta) + K^2\tilde{y}^2 = H^2$$

Compering these equations we see that curves C ~~is~~ and C_{pr} both are parabolas if $\delta = 0$, ellipses if $\delta > 0$ and hyperbolas if $\delta < 0$;

1) $\delta = 0$

$$C_{pr}: \quad \frac{2H^2}{L}x + K^2y^2 = H^2$$ parabola

$$C: \quad \frac{2H^2}{L}\tilde{x}\cos\theta + K^2\tilde{y}^2 = H^2$$ parabole.

2) $\delta > 0$

$$C_{pr}: \quad \delta\left(x + \frac{H^2}{L\delta}\right)^2 + K^2y^2 = H^2 + \frac{H^4}{L^2\delta}$$ ellipse

~~$C: \delta(\tilde{x}\cos\theta +$~~

$$C: \delta\cos^2\theta\left(\tilde{x} + \frac{H^2}{L\delta\cos\theta}\right)^2 + K^2\tilde{y}^2 = H^2 + \frac{H^4}{L^2\delta}$$ ellipse

both curves are ellipses

3) $\delta < 0$ both curves are hyperbolas.

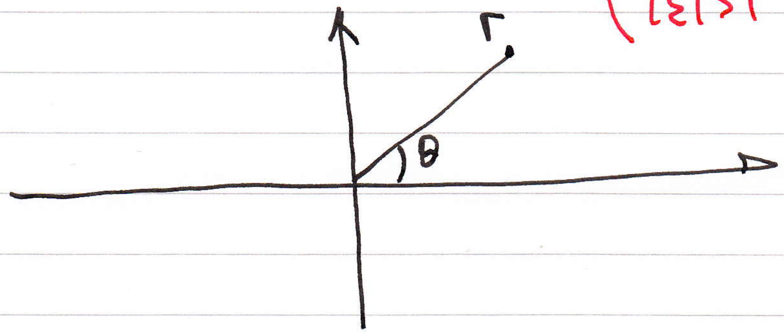
Not compulsory

[The end of the lecture is not compulsory].

Not compulsory!!!

One can show that conic section in polar coordinates is given by equation

$$r(1+\varepsilon\cos\theta) = C \qquad \left(\begin{array}{l} \varepsilon = 1 \text{ — parabola} \\ |\varepsilon|<1 \text{ — ellipse} \\ |\varepsilon|>1 \text{ — hyperbola} \end{array}\right)$$



Based on this formula one can prove the Theorem, and in particularly the fact that apex of the cone is one of foci of C_{proj}

Indeed

Let α: $ax + by + cz = 1$.

In cylindrical coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = h \end{cases}$

equation of plane α will be

$$\alpha: a r\cos\theta + b r\sin\theta + ch = 1.$$

equation of surface of cone — $\boxed{kr = h}$

Hence

$$C_{pr.}: \begin{cases} a r\cos\theta + b r\sin\theta + ch = 1 \\ h = kr \end{cases} \Longrightarrow$$

$$C_{proj} \quad r(a\cos\theta + b\sin\theta + ck) = 1.$$

$$r(\sqrt{a^2+b^2}\cos(\theta+\delta) + ck) = 1.$$

We see that C_{proj} is conic section with $F = (0,0)$ ▨