Lecture CVII

Projective transformations of $\mathbb{R}P^1$;

~~CROSS-RATIO.~~

Recall

Transformations of usual line $\mathbb{R}$

1) $u \longmapsto u + c$

translations — transformations of Euclidean line

(translations do not change 'length',
distance beetween two points)

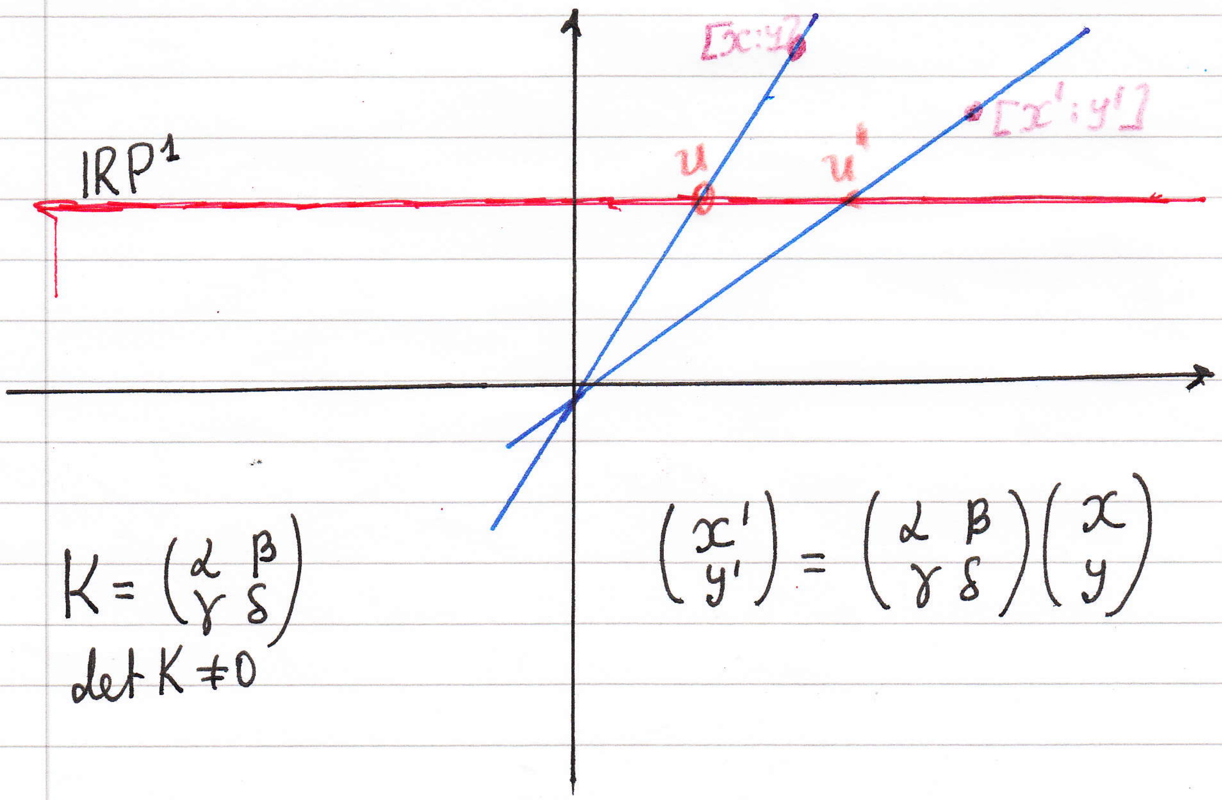
2) $u \longmapsto a u \quad (a \neq 0)$

enlargement (dilation, scaling)

$u \longmapsto a u + c$ affine transformation of $\mathbb{R}$

(transformation of affine line $\mathbb{R}$)

. . .



$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$\det K \neq 0$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

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$$x' = \alpha x + \beta y, \quad y' = \gamma x + \delta y$$

Projective transformation

$$[x:y] \longrightarrow [x':y'] = [\alpha x + \beta y : \gamma x + \delta y]$$

$$u = \frac{x}{y} \longrightarrow u' = \frac{x'}{y'} = \frac{\alpha x + \beta y}{\gamma x + \delta y}$$

$$u' = \frac{x'}{y'} = \frac{\alpha x + \beta y}{\gamma x + \delta y} = \frac{\alpha \frac{x}{y} + \beta}{\gamma \frac{x}{y} + \delta} = \frac{\alpha u + \beta}{\gamma u + \delta}$$

$$u' = \frac{\alpha u + \beta}{\gamma u + \delta}$$ – Linear fractional transformation.

Definition $$u' = \frac{\alpha u + \beta}{\gamma u + \delta} \quad (\alpha\delta - \beta\gamma \neq 0)$$

$$[x':y'] = [\alpha x + \beta y : \gamma x + \delta y]$$

projective transformation of $\mathbb{R}P^1$

Examples of projective transformations of projective line:

$$K = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$$

$$u' = \frac{3u + 2}{7u + 5}$$

$$[x':y'] = [3x + 2y : 7x + 5y]$$

$$u = 3 \longrightarrow u' = \frac{3 \cdot 3 + 2}{7 \cdot 3 + 5} = \frac{11}{26}$$

$$[x:y] = [9:3] \longrightarrow [x':y'] = [3 \cdot 9 + 2 \cdot 6 : 7 \cdot 3 + 5 \cdot 2] = [33:78]$$

$$[x':y'] = [33:78] = [11:26]$$

Lecture C VII

Take now the point at infinity. How it transforms?

$u = \{\infty\}$ $\qquad\qquad u' = \frac{x'}{y'} = \frac{3}{7}$

$[x:y] = [1:0] \longrightarrow [x':y'] = [3:7]$

We see that the point $u = \{\infty\}$, i.e. $[x:y] = [1:0]$ which is at infinity, transforms to ~~zero~~ finite point $u' = \frac{3}{7}$, i.e. $[x':y'] = [3:7]$

On the other hand

$$u = -\frac{5}{7} \longrightarrow u' = \frac{3\cdot\left(-\frac{5}{7}\right)+2}{7\cdot\left(-\frac{5}{7}\right)+5} = \frac{-\frac{1}{7}}{0} ???$$

$$[x:y] = [-5:7] \longrightarrow [x':y'] = [-15+14:0] = [-1:0]$$

$$[x':y'] = [-1:0] = [1:0], \text{ i.e. } u' = \infty$$

– Projective transformation mixes point at infinity with finite points.

Another example. $\quad K = \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix}$

$$[x':y'] = [5x+3y : y]$$

$$u' = \frac{x'}{y'} = \frac{5x+3y}{y} = \frac{5u+3}{1}$$

$$u' = 5u+3.$$

We see that translation and enlargement are special cases of projective transformations.