Lecture CVIII
CROSS-RATIO.

Projective transformation does not preserve length. What does it preserve?

Cross-ratio.

Let A, B, C, D be four distinct points on projective line

	coordinate	homogen. coordinate
A	u_A	$[x_A : y_A]$
B	u_B	$[x_B : y_B]$
C	u_C	$[x_C : y_C]$
D	u_D	$[x_D : y_D]$

Definition. Cross-ratio (A, B, C, D) of four points A, B, C, D (which are on projective line) is defined:

$$(A,B,C,D) = \frac{(u_A - u_C)(u_B - u_D)}{(u_A - u_D)(u_B - u_C)} =$$

in homogeneous coordinates

$$= \frac{\left(\frac{x_A}{y_A} - \frac{x_C}{y_C}\right)\left(\frac{x_B}{y_B} - \frac{x_D}{y_D}\right)}{\left(\frac{x_A}{y_A} - \frac{x_D}{y_D}\right)\left(\frac{x_B}{y_B} - \frac{x_C}{y_C}\right)} =$$

$$= \frac{(x_A y_C - x_C y_A)(x_B y_D - x_D y_B)}{(x_A y_D - x_D y_A)(x_B y_C - x_C y_B)} =$$

$$= \frac{\det\begin{pmatrix} x_A & x_C \\ y_A & y_C \end{pmatrix} \det\begin{pmatrix} x_B & x_D \\ y_B & y_D \end{pmatrix}}{\det\begin{pmatrix} x_A & x_D \\ y_A & y_D \end{pmatrix}\begin{pmatrix} x_B & x_C \\ y_B & y_C \end{pmatrix}}.$$

Lecture CVIII

Theorem.

Cross-ratio is invariant of projective transformations of projective line.

$$\mathbb{RP}^1 \longleftrightarrow \mathbb{RP}^1$$

$$u \longrightarrow u' = \frac{\alpha u+\beta}{\gamma u+\delta}$$

$$[x:y] \longrightarrow [x':y'] = [\alpha x+\beta y : \gamma x+\delta y]$$

$$(A,B,C,D) = (A',B',C',D')$$

Proof: straightforward calculations:

$$A \to A', \quad \begin{pmatrix} x_{A'} \\ y_{A'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x_A \\ y_A \end{pmatrix}$$

Hence

$$\det\begin{pmatrix} x_{A'} & x_{B'} \\ y_{A'} & y_{B'} \end{pmatrix} = \left[\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]^2 \det\begin{pmatrix} x_A & x_B \\ y_A & y_B \end{pmatrix}$$

$$(A',B',C',D') = \frac{\left[\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]^2 \cdot \left[\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]^2}{\left[\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]^2 \cdot \left[\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]^2} \cdot (A,B,C,D)$$

i.e.

$$(A',B',C',D') = (A,B,C,D)$$

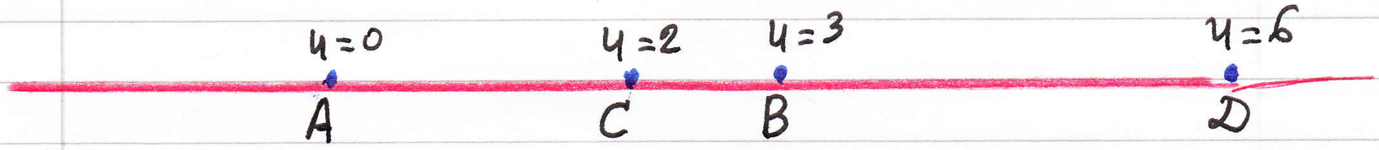
Not compulsory

Lecture CVIII

Consider four points A, B, C, D:

Cross-ratio

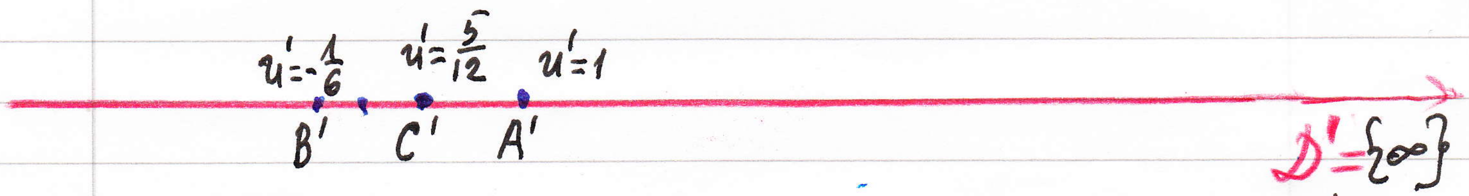
$$(A,B,C,D) = \frac{(0-2)(3-6)}{(0-6)(3-2)} = -1.$$



projective transformation

$$u' = \frac{13u-36}{6u-36}$$

$$[x':y'] = [13x-36y : 6x-36y]$$



The point A $u=0$ goes to the point A' $u'=1$, $[x':y']=[1:1]$
The point B $u=3$ goes to the point B' $u'=-\frac{1}{6}$ $[x':y']=[-1:6]$
The point C $u=2$ goes to the point C' $u'=\frac{5}{12}$ $[x':y']=[5:12]$
The point D $u=6$ goes to the point D' $u'=\infty$ $[x':y']=[1:0]$

$$(A',B',C',D') = \frac{\det\begin{pmatrix}1 & 5\\ 1 & 12\end{pmatrix}\det\begin{pmatrix}-1 & 1\\ 6 & 0\end{pmatrix}}{\det\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\det\begin{pmatrix}-1 & 5\\ 6 & 12\end{pmatrix}} = \frac{7\cdot(-6)}{(-1)(-42)} = -1$$

Cross-ratio remains the same.

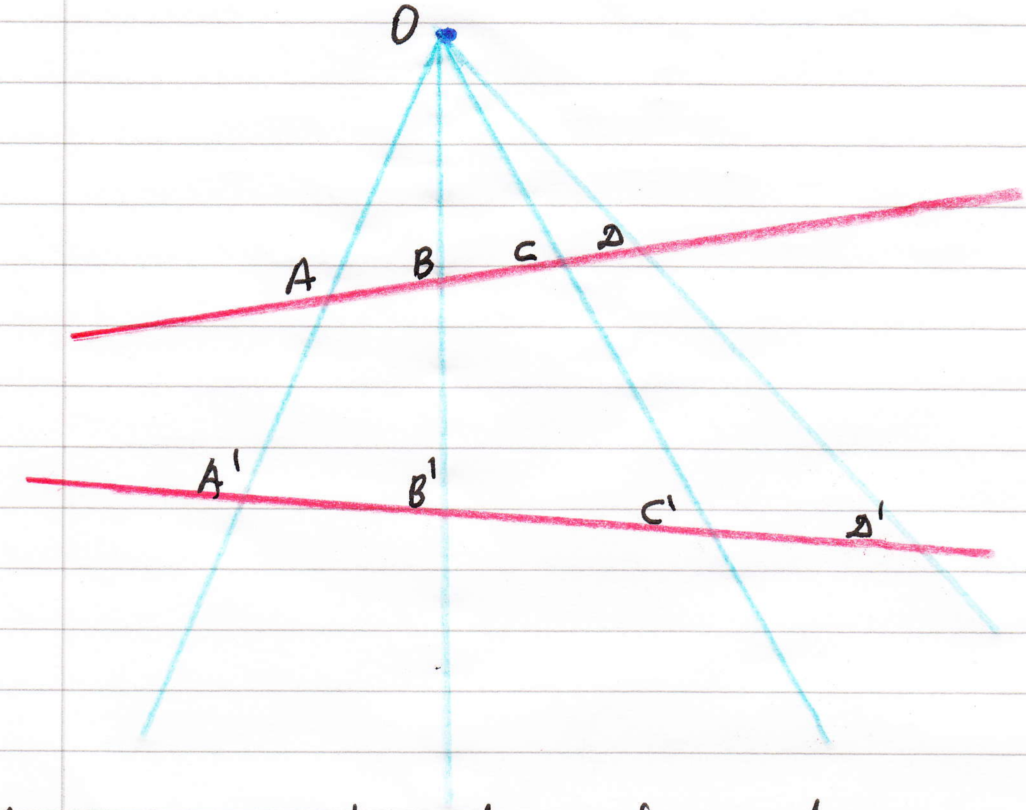
__Remark__. Cross-ratio changes if we change order of points $(A,B,C,D) \neq (B,A,C,D), \ldots$

(see Homework 9)

Lecture CVII

Where a word 'projective' comes from?

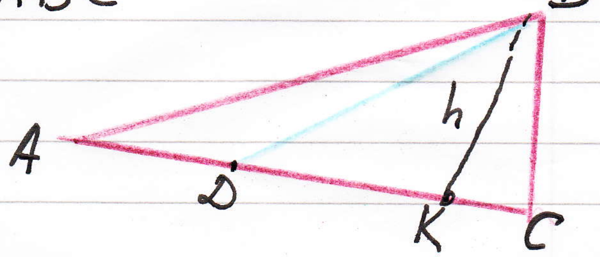
Consider central projection.



This is projective transformation.
To show it we first show that it preserves
cross-ratio: $(A, B, C, D) = (A', B', C', D')$

~~To show it consider triangles~~

Note that for arbitrary $\triangle ABC$



$$\frac{AD}{DC} = \frac{S_{\triangle ABD}}{S_{\triangle DBC}}$$

since these triangles have the same height BK

$$S_{\triangle ABD} = \frac{h \cdot AD}{2} \Rightarrow \frac{S_{\triangle ABD}}{S_{\triangle DBC}} = \frac{AD}{DC}.$$

$$S_{\triangle DBC} = \frac{h \cdot DC}{2}$$

we suppose that the plane is provided with Euclidean structure.
Our final answer will not depend on a choice of this structure.

Lecture CVII

Using this relation we will compare cross-ratios (A,B,C,D) & (A',B',C',D'):

$$(A,B,C,D)=\frac{(u_A-u_C)(u_B-u_D)}{(u_A-u_D)(u_B-u_C)}=$$

$$=\frac{u_A-u_C}{u_A-u_D}\cdot\frac{u_B-u_D}{u_B-u_C}=\frac{S_{\triangle AOC}}{S_{\triangle AOD}}\cdot\frac{S_{\triangle BOD}}{S_{\triangle BOC}}=$$

$$=\frac{|AO|\cdot|CO|\sin\measuredangle AOC}{|AO|\cdot|DO|\sin\measuredangle AOD}\cdot\frac{|BO|\cdot|DO|\sin\measuredangle BOD}{|BO|\,|CO|\sin\measuredangle BOC}=$$

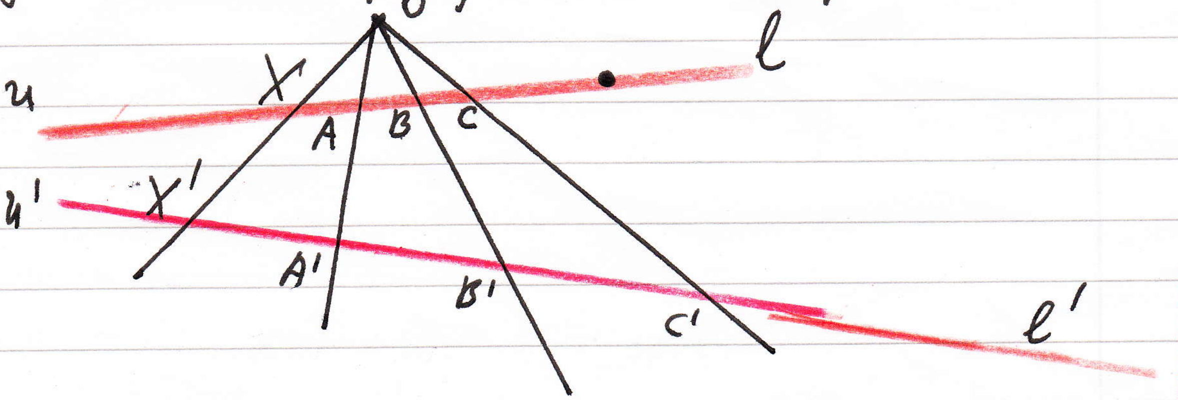
$$=\frac{\sin\measuredangle AOC\cdot\sin\measuredangle BOD}{\sin\measuredangle AOD\cdot\sin\measuredangle BOC}\qquad(*)$$

For the points A',B',C',D' we will come to $(*)$ also. Hence

$$(A,B,C,D)=(A',B',C',D').\qquad(**)$$

Sure these calculations depend on a choice of Euclidean structure [we calculated lengths, angles...] but the final relation ~~is proved~~ $(**)$ is valid irrelevant to a choice of this structure

Now we are ready to prove that central projection is projective transformation.



Choose arbitrary distinct three points A, B, C on the line l

Lecture C VIII

Let X be an arbitrary point on the line l

Let X' be its central projection on the line l'

$$(A, B, C, X) = (A', B', C', X')$$

$$\downarrow$$

$$\frac{(u_A - u_K)(u_B - u_x)}{(u_A - u_X)(u_B - u_C)} = \frac{(u'_{A'} - u'_{K'})(u'_{B'} - u'_{X'})}{(u'_{A'} - u'_{X'})(u'_{B'} - u'_{C'})}$$

$$\frac{u_B - u_X}{u_A - u_K} = R \cdot \frac{u'_{B'} - u'_{X'}}{u'_{A'} - u'_{X'}} \qquad (*)$$

where $R = \dfrac{u'_{A'} - u'_{C'}}{u'_{B'} - u'_{C'}} \cdot \dfrac{u_B - u_C}{u_A - u_C}$

Relation $(*)$ implies that

$u'_{X'}$ and u_X are related

with linear fractional transformation. $\square$

not compulsory