Lecture C X

Lines in $\mathbb{R}P^2$; Collinear points in $\mathbb{R}P^2$

point in $\mathbb{R}P^2$ — line in $\mathbb{R}^3$ passing through origin, $l \in \mathbb{R}^3$, $O \in l$.

$A \in \mathbb{R}P^2$

$[x_A : y_A : z_A]$

$$l = \begin{cases} x = t\,x_A \\ y = t\,y_A \\ z = t\,z_A \end{cases}, \quad -\infty < t < \infty$$

if $z_A \neq 0$, $u_A = \frac{x_A}{z_A}$, $V_A = \frac{y_A}{z_A}$

$$\vec{l} = t\,\vec{r}_A$$

if $z_A \neq 0$ l intersects plane Π

Line in $\mathbb{R}P^2$ — plane in $\mathbb{R}^3$ passing through origin

A, B – two points in $\mathbb{R}P^3$ — l_A, l_B two lines in $\mathbb{R}^3$

$A = [x_A : y_A : z_A]$ — $l_A = \begin{cases} x = t\,x_A \\ y = t\,y_A \\ z = t\,z_A \end{cases}, \quad -\infty < t < \infty$

$$\vec{l}_A = t\,\vec{r}_A, \quad \vec{r}_A = (x_A, y_A, z_A)$$

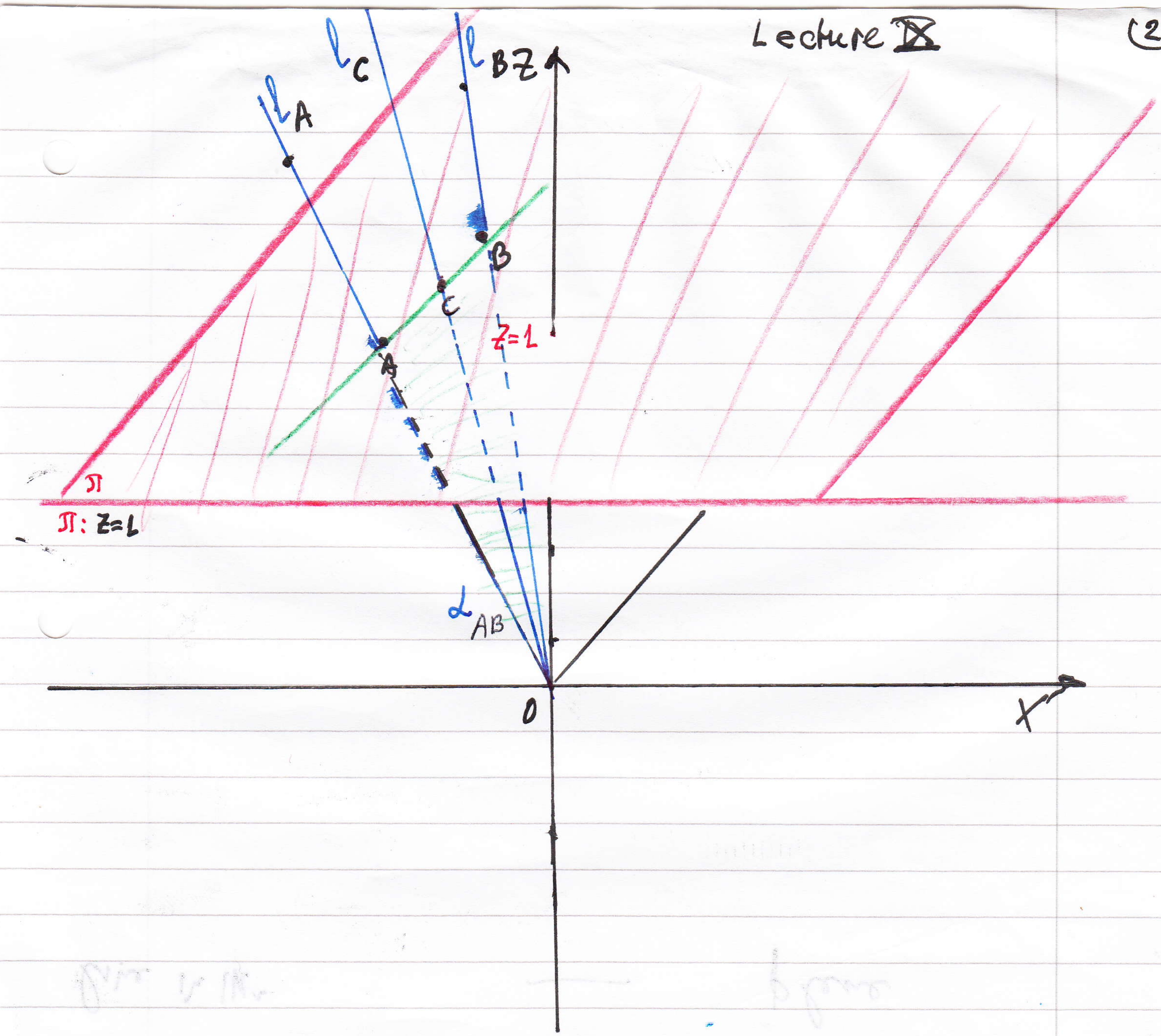
$B = [x_B : y_B : z_B]$ — $l_B = \begin{cases} x = t\,x_{AB} \\ y = t\,y_B \\ z \quad t\,z_B \end{cases}, \quad -\infty < t < \infty$

$$\vec{l}_B = t\,\vec{r}_B, \quad \vec{r}_B = (x_B, y_B, z_B)$$

Line AB passing through points A, B — Plane α_{AB} passing through origin, $\vec{r}_A$, $\vec{r}_B$

α_{AB} – span of $\vec{r}_A$, $\vec{r}_B$

$$\alpha_{AB} = \{\lambda\,\vec{r}_A + \mu\,\vec{r}_B,\ \lambda, \mu \in \mathbb{R}\}$$



$A - [x_A : y_A : z_A]$

$\vec{r}_A = (x_A, y_A, z_A)$

$\vec{l}_A = t\vec{r}_A, \ l: \begin{cases} x = t x_A \\ y = t y_A \\ z = t z_A \end{cases}, \ 0<t<\infty$

$B - [x_B : y_B : z_B]$

$\vec{r}_B = (x_B, y_B, z_B)$

$\vec{l}_B = t\vec{r}_B, \ l: \begin{cases} x = t x_B \\ y = t y_B \\ z = t z_B \end{cases}, \ 0<t<\infty$

Let C be an arbitrary point on $\mathbb{RP}^2$

Definition. We say that points A, B, C are collinear if they belong to the same line, $C \in AB$.

What is a condition, that $C \in AB$, i.e. points A, B, C are collinear?

$C - [x_c : y_c : z_c]$

$\vec{r}_c = (x_c, y_c, z_c)$

$$\vec{l}_c = t\vec{r}_c,\quad l: \begin{cases} x = t\,x_c \\ y = t\,y_c \\ z = t\,z_c \end{cases} \quad -\infty < t < \infty$$

$C \in AB$

$\vec{l}_c$ is linear combination of $\vec{l}_A$, $\vec{l}_B$

$$\vec{l}_c = m\vec{l}_A + n\vec{l}_B \qquad m, n \in \mathbb{R}$$

$$\vec{r}_c = m\vec{r}_A + n\vec{r}_B$$

$$m\begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} + n\begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix}$$

$$T_{ABC} = \begin{pmatrix} x_A & x_B & x_c \\ y_A & y_B & y_c \\ z_A & z_B & z_c \end{pmatrix}$$

Points A, B, C are collinear $(C \in AB)$

$\Updownarrow$

Matrix T_{ABC} is degenerate ($\det T_{ABC} = 0$).

Lecture C X

Suppose points A, B, C are finite points,

i.e. $z_A \neq 0$, $z_B \neq 0$

$A = [x_A : y_A : z_A]$ – affine coord. $u_A = \frac{x_A}{z_A}$, $v_A = \frac{y_A}{z_A}$

$B = [x_B : y_B : z_B]$ – affine coord, $u_B = \frac{x_B}{z_B}$, $v_B = \frac{y_B}{z_B}$

$C = [x_C : y_B : z_C]$ – affine coord, $u_C = \frac{x_C}{z_C}$, $v_C = \frac{y_C}{z_C}$

Then

A, B, C – collinear $(C \in AB)$

$$\Updownarrow$$

$$\det T_{ABC} = 0.$$

$$\Updownarrow$$

~~Points (u_A, v_A), (u_A, v_B), (~~

Points (u_A, v_A), (u_B, v_B), (u_C, v_C)

are on the same line.

Consider example.

(See in detail Exercise 5 in Homework 9).

Lecture C X

$A = [1 : -1 : 1]$, affine coordinates $u_A = 1$, $v_A = -1$

$B = [10 : -15 : 5]$, affine coordinates $u_A = 2$, $v_A = -3$

$C = [1 : -\frac{9}{5} : \frac{1}{5}]$, affine coordinates, $u_A = 5$, $v_A = -9$

Show that these points are collinear, $C \in AB$

Consider matrix

$$T_{ABC} = \begin{pmatrix} 1 & 10 & 1 \\ -1 & -15 & -\frac{9}{5} \\ 1 & 5 & \frac{1}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ -1 & -3 & -\frac{9}{5} \\ 1 & 1 & \frac{1}{5} \end{pmatrix} \sim$$

$$\sim \underbrace{\begin{pmatrix} 1 & 2 & 5 \\ -1 & -3 & -9 \\ 1 & 1 & 1 \end{pmatrix}}_{\text{determinant of this matrix vanishes}} \Longrightarrow T_{ABC}$$ is degenerate matrix, points A, B, C are collinear, i.e. $C \in AB$

another solution:

affine coordinates

$$\begin{matrix} u \\ v \end{matrix} \quad \begin{pmatrix} u_A \\ v_A \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \begin{pmatrix} u_B \\ v_B \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}; \quad \begin{pmatrix} u_C \\ v_C \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \end{pmatrix}$$

These points belong to the line $2u + v = 1$

$$2u_A + v_A = 2u_B + v_B = 2u_C + v_C = 1.$$

(See also Homework 9 (C3)).