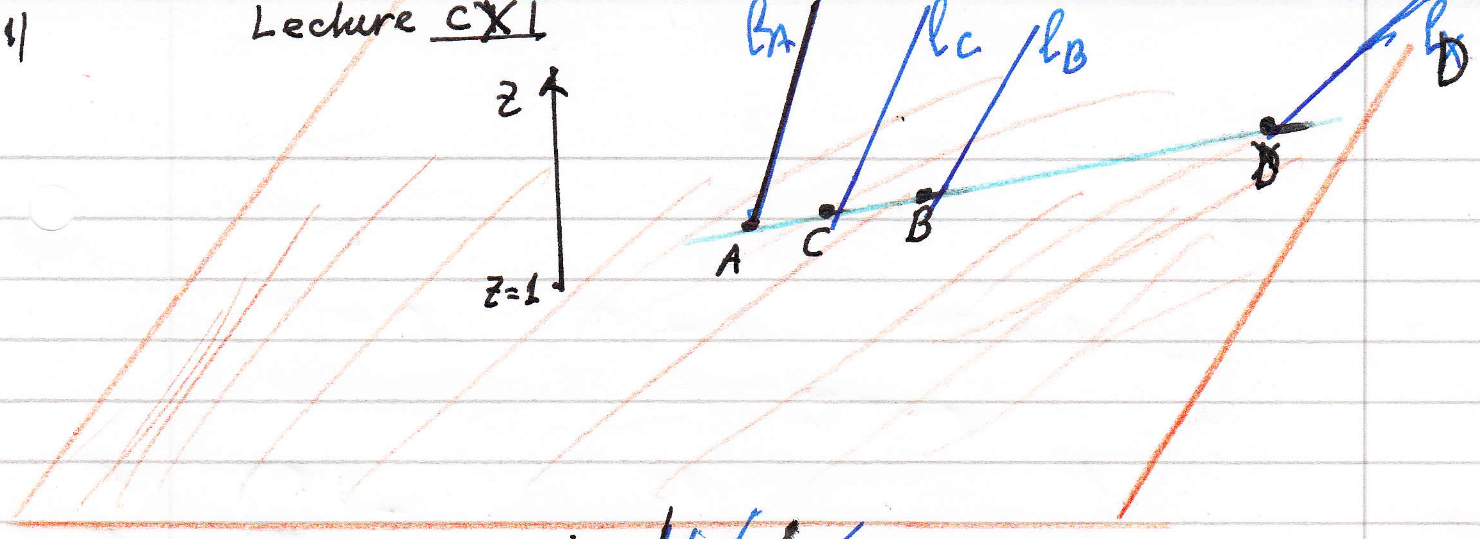
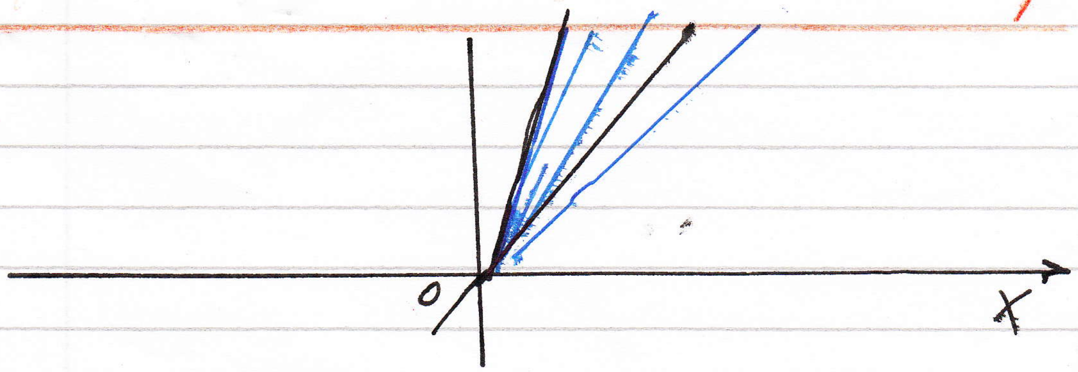
Lecture CXL





Let we have four points $A, B, C, D \in \mathbb{R}P^2$, which are collinear, i.e. they belong to the same projective line.
Calculate their cross-ratio.

homogeneous coord.	affine coordinates
$A = [8 : 20 : 4]$	$u = \frac{8}{4} = 2, \ v = \frac{20}{4} = 5$
$B = [8 : 18 : 2]$	$u = \frac{8}{2} = 4, \ v = \frac{18}{2} = 9$
$C = [6 : 14 : 2]$	$u = \frac{6}{2} = 3, \ v = \frac{14}{2} = 7$
$D = [x : 2x+1 : 1]$	$u = \frac{x}{1} = x, \ v = \frac{2x+1}{1} = 2x+1$

First check that these points are really collinear.

Lecture C XI

$$T_{ABC} = \begin{pmatrix} 8 & 8 & 6 \\ 20 & 18 & 14 \\ 4 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & 3 \\ 5 & 9 & 7 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det = 2(9-7) - 4(5-7) + 3(5-9) = 4 + 8 - 12 = 0.$$

matrix T_{ABC} is degenerate $\Longleftrightarrow$ A, B, C are collinear, $C \in AB$.

Analogously consider points B, C, D:

$$T_{BCD} = \begin{pmatrix} 8 & 6 & x \\ 18 & 14 & 2x+1 \\ 2 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & x \\ 9 & 7 & 2x+1 \\ 1 & 1 & 1 \end{pmatrix} \quad (x \neq 3)$$

$$\det = 4(7-2x-1) - 3(9-2x-1) + x(9-7) = 24 - 8x - 24 + 6x + 2x = 0$$

Matrix T_{BCD} is degenerate $\Longleftrightarrow$ B, C, D are collinear, $C \in BD$

We see that $\left.\begin{array}{l} A, B, C \text{ are collinear} \\ B, C, D \text{ are collinear} \end{array}\right\} \Rightarrow$ four points A, B, C, D are collinear.

Another way to see that these points are collinear:

$$\begin{array}{ccccc} & A & B & C & D \\ \begin{pmatrix} u \\ v \end{pmatrix} - & \begin{pmatrix} 2 \\ 5 \end{pmatrix} & \begin{pmatrix} 4 \\ 9 \end{pmatrix} & \begin{pmatrix} 3 \\ 7 \end{pmatrix} & \begin{pmatrix} x \\ 2x+1 \end{pmatrix} \end{array}$$

$u = \frac{x}{z}$, $v = \frac{y}{z}$

All these points belong the line $\boxed{2u = v - 1}$

$$v_A - 2u_A = 5 - 4 = v_B - 2u_B = 9 - 8 = v_C - 2u_C = 7 - 6 = v_D - 2u_D = 2x+1-2x = 1.$$

Lecture CXI

To calculate the cross-ratio of collinear points A, B, C, D we choose any affine coordinate*

$$(A,B,C,D) = \frac{(u_A-u_C)(u_B-u_D)}{(u_A-u_D)(u_B-u_C)} = \frac{(2-3)(4-x)}{(2-x)(4-3)}$$

$$(A,B,C,D) = \frac{x-4}{2-x}.$$

We can choose another coordinate

$$(A,B,C,D) = \frac{(V_A-V_C)(V_B-V_D)}{(V_A-V_D)(V_B-V_C)} = \frac{(5-7)(9-2x-1)}{(5-2x-1)(9-7)} = \frac{x-4}{2-x}.$$

We can choose $u' = 2u+5V \rightarrow$ . .

Cross-ratio is ~~an~~ invariant. Changing affine coordinate we will not change cross-ratio!

* * *

Now let point D 'goes' to infinity:

$$D = [x : 2x+1 : 1] = \left[1 : 2+\tfrac{1}{x} : \tfrac{1}{x}\right] \xrightarrow[x\to\infty]{} [1:2:0]$$

$D_\infty = [1:2:0]$ is a point at infinity which belongs to projective line AB

To calculate cross-ratio:

$$(A,B,C,D) = \frac{x-4}{2-x} \rightarrow -1$$

$$(A,B,C,D) = \frac{u_A-u_C}{u_B-u_C} \quad \text{if } D \text{ is at infinity}$$

(See exercise 3 in the Homework 9 (C3))

* This coordinate has to be different for different points.

not compulsory

Lecture CXI

Projective transformations of $\mathbb{R}P^2$

Recall projective transformations of $\mathbb{R}P^1$

$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $[x:y] \longrightarrow [x':y'] = [\alpha x + \beta y : \gamma x + \delta y]$

$\det K \neq 0$ $\quad u \longrightarrow u' = \frac{x'}{y'} = \frac{\alpha x + \beta y}{\gamma x + \delta y} = \frac{\alpha u + \beta}{\gamma u + \delta}$

(See the lecture CVII)

Now define projective transform. of $\mathbb{R}P^2$

$$K = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \det K \neq 0$$

$$[x:y:z] \longrightarrow [x':y':z'] = [ax+by+cz : dx+ey+fz : gx+hy+iz]$$

$$\left(u = \frac{x}{z}, \ v = \frac{y}{z}\right) \longrightarrow \left(u' = \frac{x'}{z'}, \ v' = \frac{y'}{z'}\right)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{x'}{z'} \\ \frac{y'}{z'} \end{pmatrix} = \begin{pmatrix} \frac{ax+by+cz}{gx+hy+iz} \\ \frac{dx+ey+fz}{gx+hy+iz} \end{pmatrix} = \begin{pmatrix} \frac{au+bv+c}{gu+hv+i} \\ \frac{du+ev+f}{gu+hv+i} \end{pmatrix}$$

Not compulsory

Projective transformation of $\mathbb{R}P^2$

This transformation transforms a (projective) line to a (projective) line.

It includes usual affine transformations

Lecture CXI

Indeed consider

$$K = \begin{pmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{pmatrix}$$

We have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{au+bv+p}{0\cdot u+0\cdot v+1} \\ \frac{cu+dv+q}{0\cdot u+0\cdot v+1} \end{pmatrix} = \begin{pmatrix} au+bv+p \\ cu+dv+q \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$$

This usual affine transformation.

Projective transf. transform points (of $\mathbb{RP}^2$) to points (of $\mathbb{RP}^2$) lines (proj) to lines (proj.)

Cross-ratio is invariant of projective transform.

Affine transformation is a transform. of projective plane which does not move projective line at infinity.

NOT COMPULSORY