Lecture XII

Return to conic sections

Recall that linear transformation $\begin{cases} x = ax' \\ y = by' \end{cases}$ $(*)$

transforms ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

to circle $(x')^2 + (y')^2 = 1$.

[Sure $(*)$ is not orthogonal transformation (if $a \neq 1, b \neq 1$)]

Consider the following example

$C: \quad x^2 + y^2 + 2pxy + x + y = 1$ in $\mathbb{R}^2$

How looks this curve?

[We will allow not only isometries (orthogonal transformations and translations) but also arbitrary affine transformations*

Consider first linear tranformation

$\begin{cases} x = u + v \\ y = u - v \end{cases}$ This is linear transformation, which is not orthogonal. — It does not preserve the length and scalar product [This is not isometry]

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* Moreover we will allow in the second part of lecture also projective transformations which are not affine.

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$C: \quad x^2 + y^2 + 2pxy + x + y = 1$

$$\Big\updownarrow \quad \begin{cases} x = u + v \\ y = u - v \end{cases}$$

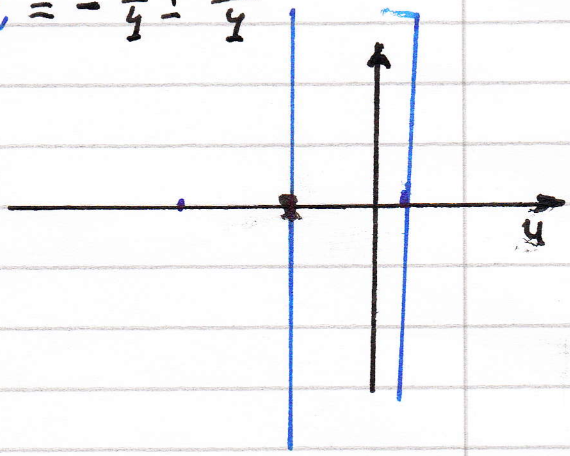
$(u+v)^2 + (u-v)^2 + 2p(u+v)(u-v) + 2u = 1.$

$2(1+p)u^2 + 2(1-p)v^2 + 2u = 1.$

Consider cases.

1) $p = 1.$ $\quad 4u^2 + 2u = 1. \quad u = -\frac{1}{4} \pm \frac{\sqrt{5}}{4}$

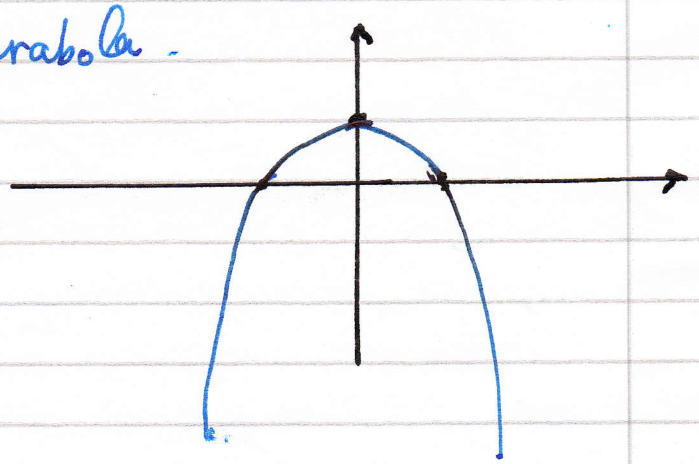
C: two vertical lines



2) $p = -1$

$4v^2 + 2u = 1$

parabola.



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3) $-1<p<1$

$$2(1+p)u^2+2u+2(1-p)v^2=1$$

$$2(1+p)\left[u+\frac{1}{2(1+p)}\right]^2+2(1-p)v^2=1+\frac{1}{2(1+p)}. \qquad (*)$$

This is an ellipse with centre at the point $u=-\frac{1}{2(1+p)}$, $v=0$.

Under ~~linear~~ affine transformation

$$\begin{cases} u = -\frac{1}{2(1+p)} + C_1\tilde{u} \\ v = C_2\tilde{v} \end{cases} \qquad (**)$$

with specially chosen C_1, C_2

it will be transformed to the circle

$$\boxed{\tilde{u}^2+\tilde{v}^2=1.}$$

4) $p>1$ or $p<-1$

In this case $(*)$ defines hyperbola

Under ~~linear~~ affine transform. $(**)$ (with especially chosen C_1, C_2) it will transform to hyperbola $\tilde{u}^2-\tilde{v}^2=1$

Not compulsory

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NOT COMPULSORY

What will happen if we allow not only arbitrary affine transformations, but also projective?

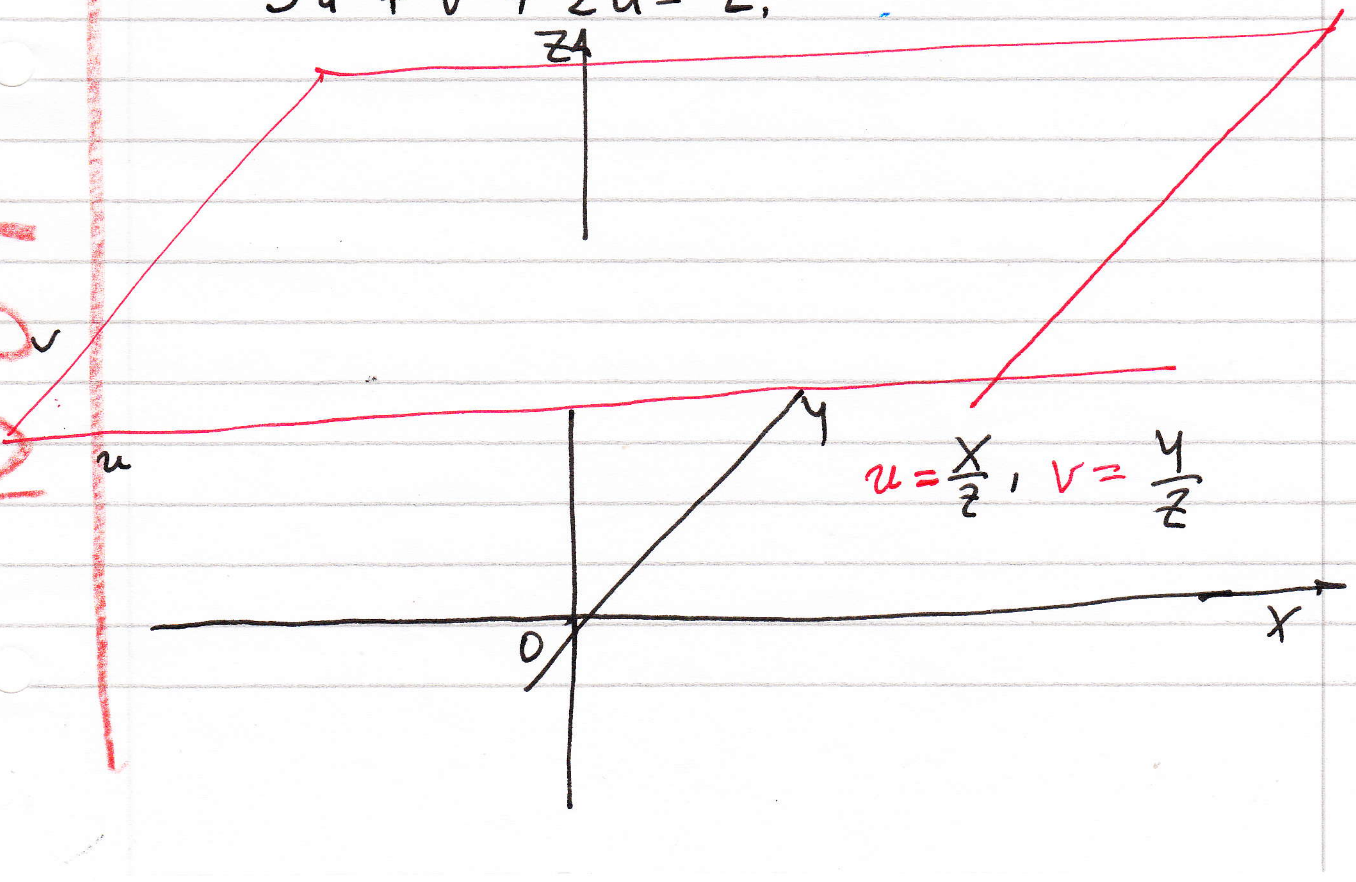
Show that the curve

$$2(1+p)u^2+2(1-p)v^2+2u=1.$$

can be transformed to circle by projective transformations, (except degenerate case $p=1$), i.e. parabola, hyperbole and ellipse do not differ ~~fi~~ in projective geometry.

Example. Consider $p=\frac{1}{2}$

$$3u^2+v^2+2u=1.$$



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$$3u^2 + v^2 + 2u = 1 \qquad (*)$$

$$\Updownarrow$$

$$\frac{3X^2}{Z^2} + \frac{Y^2}{Z^2} + 2\frac{X}{Z} = 1$$

$$3X^2 + Y^2 + 2XZ - Z^2 = 0$$

Projective transformation.

$$[X' : Y' : Z'] = [Z : Y : X]$$

$$\Updownarrow$$

$$3Z^2 + Y^2 + 2XZ - X^2 = 0$$

$$u' = \frac{X'}{Z'} \qquad v' = \frac{Y'}{Z'}$$

$$\underline{3 + v^2 + 2u - u^2 = 0} \qquad (**)$$

Hyperbola.

NOT COMPULSORY

Ellipse(*) becomes Hyperbola.(**) under projective transformation

$$[X : Y : Z] \longrightarrow [Z : Y : X]$$

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NOT COMPULSORY

Example

$u^2+v^2=1.$

$$X^2+Y^2=Z^2 \quad \left(u=\frac{X}{Z},\ v=\frac{Y}{Z}\right)$$

$$[X:Y:Z] \longleftrightarrow [Z:Y:X]$$

$$Z^2+Y^2=X^2$$

$u^2-v^2=1 \longrightarrow$ hyperbola.

$$[X:Y:Z] \longleftrightarrow [X+Z:Y:X-Z]$$

$$X^2+Y^2=Z^2$$

$$\updownarrow$$

$$(X+Z)^2+Y^2=(X-Z)^2$$

$$4XZ+Y^2=0.$$

$4u+v^2=0 \longrightarrow$ parabola

$\left(u=\frac{X}{Z},\ v=\frac{Y}{Z}\right).$

PARABOLA

Ellipse

Hyperbola

are on an equal footing in projective geometry.