Introduction to Geometry (20222)

COURSEWORK 2019

Discussions

`Here we discuss the coursework. (The solutions of coursework problems with returned courseworks are in the Reception)`

11 April 2019

1

a) *Let* (x^1, x^2) *be coordinates of the vector* $\mathbf{x}$*, and* (y^1, y^2) *be coordinates of the vector* $\mathbf{y}$ *in* $\mathbf{R}^2$.

Consider the formula

$$(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + kx^1y^2 + kx^2y^1 \,, \tag{1.1}$$

where k *is a real parameter. Show that this formula defines a scalar product in* $\mathbf{R}^2$ *in the case if* $|k| < 1$.

Give an example of orthonormal basis for this scalar product.

Explain why this formula does not define a scalar product on $\mathbf{R}^2$ *in the case if* $|k| \geq 1$.

b) *Consider the matrix* $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Calculate the matrix A^9 *in the case if* $\theta = \frac{\pi}{27}$.

Calculate the matrix A^{2019} *in the case if* $\theta = \frac{\pi}{6}$.

c) *In Euclidean space* $\mathbf{E}^3$ *consider the following linear operator*

$$A(\mathbf{x}) = \mathbf{x} + (\mathbf{a}, \mathbf{x})\mathbf{a}$$

where the vector $\mathbf{a} = 3\mathbf{e} + 4\mathbf{f} + 12\mathbf{g}$. *Here* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is an orthonormal basis in* $\mathbf{E}^3$.

Find eigenvalues and eigenvectors of operator A.

Calculate the trace and determinant of the operator A.

d) *Let* $\{\mathbf{e}, \mathbf{f}\}$ *be an orthonormal basis of Euclidean space* $\mathbf{E}^2$. *Consider a linear operator* P *such that* $\mathbf{a} = P(\mathbf{e}) = 91\mathbf{e} + 50\mathbf{f}$, $\mathbf{b} = P(\mathbf{f}) = 20\mathbf{e} + 11\mathbf{f}$.

Calculate determinant of the operator P.

Show that P *is not an orthogonal operator.*

Does this operator preserve ab orientation of $\mathbf{E}^2$ *? Justify you ranswer.*

Consider the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$ attached at the origin. Find the area of this parallelogram.

Show that the vertices of the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ are the only points of $\Pi_{\mathbf{a},\mathbf{b}}$, whose coordinates are both integers.

a)

The condition of linearity and symmetricity for 'scalar product' can be checked almost automatically. A question is to check the condition of positive defniteness, i.e. the condition that $B(\mathbf{x},\mathbf{x}) > 0 \Leftrightarrow \mathbf{x} \neq 0$. Almost all students had problem to explain clearly and shortly why the condition $|k| < 1$ is necessary and sufficient condition for positive definitness.

Recall (see solutions) that this immediately follows from the relation

$$B(\mathbf{x},\mathbf{x}) = \left(x^1\right)^2 + \left(x^2\right)^2 + 2kx^1x^2 = \left(x^1 + kx^2\right)^2 + (1-k^2)\left(x^2\right)^2 .$$

Only 10-12 students wrote the orthonormal basis for this scalar product (in the case if $|k| < 1$).

Many students insisted that vectors $\mathbf{e} = (1,0), \mathbf{f} = (0,1)$ form orthonormal bbasis, in spite of the fact that this is obviously not true: $(\mathbf{e},\mathbf{f}) = k \neq 0$.

b) Students have no problems to solve this question

c) Almost ALL students (except two of them) were solving this question straightforwardly: they wrote the matrix $A =$ of the operator in the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ and straightforwardly caclulated determinant and trace of this 3×3 matrix. Almost nobody made mistakes during these calculations, however all students (except two) missed beautifual and short solution: it is evident that the vector $\mathbf{a}$ and vectors orthogonal to this vector are eigenvectors of operatrot A: $A(\mathbf{x}) = \mathbf{x}$ for every vector $\mathbf{x}$ which is rothogonal to the vector $\mathbf{a}$, and $A(\mathbf{a}) = (1 + (\mathbf{a},\mathbf{a}))\,\mathbf{a} = ((1 + (9+16+144))\,\mathbf{a} = 170\mathbf{a}$. Hence

$$\det A = 1 \cdot 1 \cdot 170 = 10$$

and

$$\operatorname{Tr} A = 1 + 1 + 170 = 172 .$$

This is nice is not it?

I am happy that two students came to this solution.

d) Almost all students calculated the determinant of the operator P and showed that P is not orthogonal operator, however many (too many!?) students did it using brute force and calculating straightforwardly the matrix of operator $P^T \circ P$, which possesses big numbers:

$$P^T \circ P = \begin{pmatrix} 91 & 20 \\ 50 & 11 \end{pmatrix} \begin{pmatrix} 91 & 50 \\ 20 & 11 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

However to see that P is not orthogonal operator you do not need to perform these calculations. E.g. the fact that P is not orthogonal operator follows from the observation that the length of the vector $\mathbf{a}$ is not equal to 1 (see in details solution)

The last part of this question was difficult one.

Many students were trying to do the right considerations, and few students came to the right solution. I will present here two nice geometrical solutions.

First solution Consider the parallelogram $\Pi_{\mathbf{e},\mathbf{f}}$ formed by basis vectors. This parallelogram is a unit square. It obviously does not possess integer points except vertices. The linear operator P with matrix transforms the square $\Pi_{\mathbf{e},\mathbf{f}}$ onto the paralellogram $\Pi_{\mathbf{a},\mathbf{b}}$. The inverse operator P^{-1} transforms the parallelogram $\Pi_{\mathbf{a},b}$ onto the square $\Pi_{\mathbf{e}_x,\mathbf{e}_y}$. The idea of the proof is the following: the matrix of the operator P and the matrix of the inverse operator P^{-1} in the basis $\mathbf{e},\mathbf{f}$ have integer entries. This implies that all integer points of the unit square are in one-one correspondence with integer points of the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$, and the unit square has no integer points in it except vertices.

Second solution The proof follows from

Lemma Any triangle with vertices in integer points has area equal or bigger than $\frac{1}{2}$.

Proof of the lemma: Let triangle be formed by two vectors $\mathbf{c},\mathbf{d}$. Take the parallelogram $\Pi_{\mathbf{c},\mathbf{d}}$ corresponding to this triangle. This parallelogram has vertices in integer points. Hence by determinant formula its area is bigger or equal to 1. Hence the area of triangle is bigger or equal to 1/2.

Now based on the lemma prove that the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ has no a point with integer coordinates except vertices. Take any point A in this parallelogram. In the case if A does not coincide with one of vertices, then one can form at least three triangles in this parallelogram which do not interesect (and even 4 triangles if this point is an interior point). Suppose that A has integer coefficients. Then by lemma we see that area of parallelogram is bigger or equal than $3 \cdot \frac{1}{2} = \frac{3}{2} > 1$. Contradiction with the fact that parallelogram has an area 1 [1)].

2

We consider in this question 3-dimensional Euclidean space $\mathbf{E}^3$. We suppose that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in this space.

a) *Let P be a linear orthogonal operator acting in $\mathbf{E}^3$ such that its matrix in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ has the following appearance*

$$P = \frac{1}{7}\begin{pmatrix} 3 & * & 6 \\ -6 & -3 & 2 \\ 2 & -6 & * \end{pmatrix} .$$

*Find the entries of the matrix denoted by *.*

Show that the operator P preserves orientation.

We know that due to the Euler Theorem the linear operator P considered above is a rotation operator. Find the axis and the angle of this rotation.

b) *Let P_1 be a rotation operator on the angle θ around the axis directed along the vector $\mathbf{g}$, and P_2 be a rotation operator on the same angle θ around the axis directed along the vector $\mathbf{e}$:*

$$\{\mathbf{e},\, \mathbf{f},\, \mathbf{g}\} \xrightarrow{P_1} \{\cos\theta\mathbf{e} + \sin\theta\mathbf{f},\ -\sin\theta\mathbf{e} + \cos\theta\mathbf{f},\ \mathbf{g}\}\,,$$

$$\{\mathbf{e},\, \mathbf{f},\, \mathbf{g}\} \xrightarrow{P_2} \{\mathbf{e},\ \cos\theta\mathbf{f} + \sin\theta\mathbf{g},\ -\sin\theta\mathbf{f} + \cos\theta\mathbf{g}\}\,.$$

[1)] There are another beautiful proofs of this fact. It has to be mentioned that all this stuff is related with continuous fractions. and the Pick formula that states that any convex polygon with vertices in integer points has the area

$$S = \frac{E}{2} + I - 1\,,$$

where E is a number of points which belong to edges (including vertices), and I the number of points which belong to interior of the polygon. (In fact we are on the way to prove the Pick formula).

Show that the operator $P = P_1 \circ P_2$ is also a rotation operator. Find the axis of rotation and the angle $\Phi = \Phi(\theta)$ of rotation for the operator P.

Calculate the angle $\Phi = \Phi(\theta)$ in the case $\theta = \frac{\pi}{2}$.

Show that in the case if θ is small, then $\Phi(\theta) \approx \sqrt{2}\theta$, i.e.

$$\lim_{\theta \to 0} \frac{\Phi(\theta)}{\theta} = \sqrt{2}\,.$$

2

Almost nobody had problems to answer the question 2a)

Answering question 2b) students had no problems to calculate the matrix P of the operator $P_1 \cdot P_2$

Some students when proving the fact that $P_1 \circ P_2$ is also orthogonal operator preserving orientation did it using brute force: they just calculated straightforwardly that $P^T \cdot P = \mathbf{id}$ and $\det P = 1$ for matrix P of operator $P_1 \circ P_2$. Instead doing these calculations one can deduce it from the properties of operators P_1, P_2 or corresponidng matrices. E.g. an operator $P = P_1 \circ P_2$ is orthogonal because it is product of two orthogonal operators:

$$P^T \cdot P = (P_1 \cdot P_2)^T \cdot (P_1 \cdot P_2) = P_2^T \cdot (P_1^T \cdot P_1) \cdot P_2 = P_2^T \cdot P_2 = \mathbf{id}$$

and its determionant is equal to 1 because

$$\det P = \det(P_1 \cdot P_2) = \det P_1 \det P_2 = 1 \cdot 1 = 1\,.$$

One does not need to do straightworward calculations of determinant of the matrix P.

Some students came to the conclusion that $P = P_1 \cdot P_2$ is rotation operator just on the base that this operator preserves orientation, This is not enough: operator P is rotation operator since

1) *it preserves orientation*

2) *and it is orthogonal operator.*

Both conditions have to be checked.

This is a mistake to think that $\det P = 1$ implies that P is orthogonal operator. Please avoid it!

Some students just ignored to calculate the axis of rotation operator P.

The answer on the last question: that for small θ, $\Phi \approx \sqrt{2}\theta$ can be done by study of the formula

$$\cos\Phi = \cos\theta + \frac{1}{2}(\cos^2\theta - 1)\,. \qquad (*)$$

Many students came the answer, but only few analyzed it properly. Few students were trying to use L'Hopital's rule. Using this rule one has to be carreful with the fact that derivative of the function arccosx ($\cos^{-1} x$) at the point $x = 1$ is not well-defined.

Recall that the short and clear solutions avoiding the long calculations follows from the formula

$$\cos\theta = 1 - \frac{\theta^2}{2} + o(\theta^2)$$

(see solutions).

I wold like to show another simple and clear solution, which does not use much calculus: We have to solve equation

$$\Phi(\theta): \quad \cos\Phi = \cos\theta - \frac{1}{2}\sin^2\theta\,.$$

Transform it:

$$\cos\Phi = 1 - 2\sin^2\frac{\Phi}{2} = \cos\theta - \frac{1}{2}\sin^2\theta = 1 - 2\sin^2\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\,.$$

i.e.

$$\sin^2\frac{\Phi}{2} = \left(1 + \cos^2\frac{\theta}{2}\right)\sin^2\frac{\theta}{2} = \left(2 - \sin^2\frac{\theta}{2}\right)\sin^2\frac{\theta}{2}\,,$$

and

$$\Phi = 2\text{arcsin}\left(\sqrt{2 - \frac{\sin^2\frac{\theta}{2}}{2}}\sin\frac{\theta}{2}\right) = 2\cdot\sqrt{2}\frac{1}{2}\theta + 0(\theta) = \sqrt{2}\theta + o(\theta)\,.$$

Remark The geometrical interpretation of this result is the following: If angle θ is very small, then infinitesimally, action of rotation operator is $P(\mathbf{x}) = \mathbf{x} + \theta\mathbf{w}\times\mathbf{x}$, where $\mathbf{w}$ is the vector of angular velocity. Hence the result of infinitesimal rotations around axis $\mathbf{w}_1$ and $\mathbf{w}_2$ is the rotation around axis $\mathbf{w}_1 + \mathbf{w}_2$, and the length of the vector $\mathbf{e}_x + \mathbf{e}_z$ is equal to $\sqrt{2}$.

Angular velocity is the vector, two infinitesimal rotations are described by the sum of two vectors of angular velocity!

Unfortunately nobody even tried to do this.

3

a) *Consider the curve* $\mathbf{r}(t)$: $\begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases}, \quad 0 \le t \le 1.$

Draw the image of this curve.

Give an example of a parameterisation of this curve with opposite orientation.

b) *Let* f *be a function in* $\mathbf{E}^2$ *given by* $f = r^2 \cos 2\varphi$, *where* r, φ *are polar coordinates in* $\mathbf{E}^2$ ($x = r\cos\varphi, y = r\sin\varphi$). *Consider vector fields which are given in Cartesian coordinates by* $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$.

Calculate $\partial_{\mathbf{A}} f$, $\partial_{\mathbf{B}} f$.

Let g *be a function on* $\mathbf{E}^2$ *such that differential form* $\omega = df$ *vanishes at the vector field* $\mathbf{B} = x\partial_y - y\partial_x$: $\omega(\mathbf{B}) \equiv 0$. *Find a function* g *if it is known that*

$$g(x,y)\big|_{y=0} = x^6 .$$

a) Almost all studnets answerd this question right.

b)

One can calculate directional derivatives $\partial_{\mathbf{A}} f, \partial_{\mathbf{B}} f$ in Cartesian as well in polar coordinates. It is very illuminating to perform calculations in polar coordinates, still for vector fields **A** and **B** have the following appearance in poalr coordinates:

$$\mathbf{A} = r\frac{\partial}{\partial r}, \quad \mathbf{B} = \frac{\partial}{\partial \varphi}$$

This comment is especially improtant for finding a function g which is invariant with respect to the field **B**, i.e. $\partial_g \equiv 0$ and which obeys boundary condition $g(x,y)\big|_{y=0} = x^6$.

The condition $\partial_{\mathbf{B}} g = 0$ in polar coordinates implies that function g does not depend on φ, hence $g = r^6 = (x^2+y^2)^3$. We come so quick to the nice answer since we use polar coordinates.