**Two hours**

**THE UNIVERSITY OF MANCHESTER**

INTRODUCTION TO GEOMETRY

19 May 2017

XX:00 – XX:00

Answer **ALL FIVE** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (30 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

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Electronic calculators may not be used.

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**EXAM FEEDBACK**

## SECTION A

Answer **ALL** FIVE questions

**A1.**

(a) *Explain what is meant by saying that two bases in* $\mathbf{E}^3$ *have the same orientation.*

(b) *Let* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *be a basis in* $\mathbf{E}^3$*.*

*Consider the ordered triple* $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$*.*

*Show that this triple is a basis.*

*Show that the bases* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *and* $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$ *have opposite orientations.*

(c) *Let an ordered triple* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *be a basis in* $\mathbf{E}^3$*.*

*Explain why either the bases* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *and* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *have the same orientation, or the bases* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *and* $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$ *have the same orientation.*

[10 marks]

Almost all students have no problems answering this question. Two bases have the same orientation if determinant of transition matrix is positive. Almost all students answered this question properly.

The new triple $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$. is a basis because the transition matrix is non-degenerate, i.e. its determinant is not equal to zero, and the bases have opposite orientation since determinant of the transition matirx is negative. Another way to show that the new triple is a basis, (not the shortest one) is just to check straightforwardly that vectors $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$ are linearly independent. in $\mathbf{E}^3$. Some students did it in this way.

About 10 students called the transition matrix a linear operator. Marks were not decreased for this, but this is wrong! Transition matrix from one basis to another *is not a linear operator*, this is just a matrix; the entries of this matrix define the transition from one basis to another.

**A2.**

(a) *State the Euler Theorem about rotations.*

(b) *Let* $P$ *be an orthogonal operator preserving orientation such that, in an orthonormal basis* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$*,* $P$ *has the following appearance*

$$P = \begin{pmatrix} 0 & * & 0 \\ 0 & * & -1 \\ 1 & * & 0 \end{pmatrix}.$$

*Calculate the entries of this matrix which are denoted by* $*$*.*

(c) *Euler's Theorem states that the operator $P$ defines a rotation. Calculate the angle of this rotation.*

[10 marks]

The Euler Theorem on rotation states:

*Let $P$ be an orthogonal operator preserving an orientation of Euclidean space $\mathbf{E}^3$, i.e. operator $P$ preserves the scalar product and orientation. ($(P\mathbf{x}, P\mathbf{y}) = (\mathbf{x}, \mathbf{y})$, $\det P > 0$ (in fact $\det P = 1$).) Then it is a rotation operator with respect to an axis $l$ on the angle $\varphi$. Every vector $\mathbf{N}$ directed along the axis is not changed, i.e. the axis is 1-dimensional space of eigenvectors with eigenvalue 1, $P(\mathbf{N}) = \mathbf{N}$. Every vector orthogonal to axis rotates on the angle $\varphi$ in the plane orthogonal to the axis and $\operatorname{Tr} P = 1 + 2\cos\varphi$. (The angle $\varphi$ is defined up to a sign.)*

Formulating of Euler Theorem many students gave non-complete formulation. (the role of axis, the rotation of the vectors in orthogonal plane).

Few students claim the Theorem for arbitrary $n$-dimesnional Euclidean space. In fact Euler theorem is wrong for $n > 3$.

The question about reconstructing the matrix was alright, almost all students did it using orthogonal matrices. I am very happy that few students did it using just properties of linear orthogonal operators. I am very happy also that this year almost nobody did mistake confusing orthogonal matrices with unimodular matrices (matrices with unit determinant).

To calculate the angle $\alpha$ of rotation, we see that TrP $= 1 + 2\cos\alpha = 0$, hence $\cos\alpha = -\frac{1}{2}$. This immediately implies that the angle $\alpha = \pm\frac{2\pi}{3}$.

Almost all students calculated correctly that $\cos\alpha = -\frac{1}{2}$, and this is very sad that *almost all students* failed to write the ‘ answer for the angle $\alpha$. If you do not remember it you can immediately to see it just drawing the equilateral triangle $\triangle ABC$: all the angles of this triangle are equal to $\frac{\pi}{3}$ and

$$\cos\alpha = \cos\frac{\pi}{3} = \cos\angle ABC = \frac{|BC|/2}{|AB|} = \frac{1}{2}\,, \text{hence} \cos(\pi - \alpha) = -\cos\alpha = -\frac{1}{2}\,.$$

**A3.**

(a) *Give a definition of a differential 1-form in $\mathbf{E}^n$.*

(b) *Calculate the value of the 1-form $\omega = xdy - ydx$ on the vector field $\mathbf{A} = y\partial_y - x\partial_x$.*

(c) *Explain what is meant that a differential 1-form is exact. Give an example of an exact 1-form and give an example of a non-exact 1-form. Justify your answers.*

[10 marks]

Definition od differential form

*A differential* 1*-form* $\omega$ *on* $\mathbf{E}^n$ *is a function on tangent vectors of* $\mathbf{E}^n$*, such that it is linear at each point:* $\omega(\mathbf{r}, \lambda\mathbf{v}_1 + \mu\mathbf{v}_2) = \lambda\omega(\mathbf{r}, \mathbf{v}_1) + \mu\omega(\mathbf{r}, \mathbf{v}_2)$, $\lambda, \mu \in \mathbf{R}$. *Here* $\mathbf{v}_1, \mathbf{v}_2$ *are vectors tangent to* $\mathbf{E}^n$ *at the point* $\mathbf{r}$*,* $(\mathbf{v}_1, \mathbf{v}_2 \in T_x\mathbf{E}^n)$*.*

Almost all students gave the definition correctly, and caluclated the value of the form on the vector field.

Answering the second part of this question students have no problems with definition and examples of exact forms.

Now about example of non-exact form.

E.g. the form $\omega = xdy$ is not exact. Why? Suppose it is exact, i.e. $\omega = xdy = df = f_x dx + f_y dy$. We see that $f_x = \frac{\partial f(x,y)}{\partial x} = 0$ and $f_y = \frac{\partial f(x,y)}{\partial y} = x$.This immediately leads to contradiction: taking the second derivative we see that

$$f_{yx} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = 0\,, \text{and } f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = 1\,, i.e. 0 = f_{yx} = f_{xy} = 1\,. \quad (A3.1)$$

Almost all students gave the correct example of non-exact form which was usually similar to the example above, but many students have trouble justifying the example, and did not receive the full mark. Many students considering this (or similar) example have chosen another way to come to contradiction, amd did not finish the considerations. Consider typical not complete solution:

Show that $\omega = xdy$ is not exact. Suppose as above that $\omega = xdy = df = f_x dx + f_y dy$. Hence $f_x = \frac{\partial f(x,y)}{\partial x} = 0$ and $f_y = \frac{\partial f(x,y)}{\partial y} = x$. At this step instead considerations (A3.1) one can see that condition $f_x = 0$, implies that $f = \int f_x dx = g(y)$, and on the other hand condition $f_y = x$, implies that $f = \int f_y dy = xy + r(x)$, i.e.

$$f = g(y) = xy + r(x)\,, \quad (A3.2)$$

where $g$ is an arbitrary (differentiable) function on $y$ and $r$ is an arbitrary (differentiable) function on $x$. Relation (A3.2) possesses contradiction.

Yes, this is right, but it has to be justified. It can be done, for example, in the following way: differentiate relation (A3.2) with respect to $y$, we come to

$$\frac{\partial g(y)}{\partial y} = x \quad (A3.3)$$

Right hand side of this expression is the function on $x$, and the left hand side is the function on $y$. Take two distinct values of $x$, $x_1, x_2$, and an arbitrary value of $y$, $y_1$. We come to contradiction: $g'(y)$ at the point $y = y_1$ is equal to $x_1$, on the other hand $g'(y)$ at the point $y_1$ is equal to $x_2$. Contradiction.

Sure the consideration (A3.1) is much shorter than (A3.2), (A3.3).

**A4.**

*(a) Give the definition of a hyperbola with foci at the given points $F_1, F_2$.*

*(b) Let $H$ be a hyperbola in the plane $\mathbf{E}^2$ such that it passes through the point $(3,8)$, and has foci $F_1 = (3,0)$ and $F_2 = (-3,0)$.*

*Show that this hyperbola intersects the horizontal axis $OX$ at the points $(1,0)$ and $(-1,0)$.*

*(c) Explain why this hyperbola does not intersect the vertical axis $OY$.*

[10 marks]

The hyperbola $H$ with the foci $F_1, F_2$ is the locus of the points on the plane such that the difference of the distances to foci is constant:

$$H = \{K\colon\ |d(K,F_1) - d(K,F_2)| = \text{constant}\}\,. \tag{A4.1}$$

The fact that the point $K = (3,8)$ implies that

$$|d((3,8),(3,0)) - d((3,8),(-3,0))| = \left||8-0| - \sqrt{(3-(-3))^2+8^2}\right| = |8-10| = 2\,,$$

i.e. constant in relation (A4.1) is equal to 2. This immediately implies that points $(\pm 1, 0)$ belong to the hyperbola $H$:

$$|d((1,0),(3,0)) - d((1,0),(-3,0))| = ||1-3| - |1-(-3)|| = ||-2| - |4|| = |2-4| = |-2| = 2\,,$$

and

$$|d((-1,0),(3,0)) - d((-1,0),(-3,0))| = ||-1-3| - |-1-(-3)|| = ||-4| - |2|| = |4-2| == 2\,, \tag{A4.2}$$

On the other hand one can see that any point $M = (0,t)$ on the axis $0Y$ *does not belong* to the hyperbola. Indeed for $K = (0,t)$ the distances $d(K,F_1)$ and $d(K,F_2)$ coincide: $d(K,F_1) = d(K,F_2) = \sqrt{t^2+9}$,

$$\text{for } K = (0,t)\ |d(K,F_1) - d(K,F_2)| = 0 \neq 2 \tag{A4.3}$$

Almost all students gave geometrical definition of hyperbola correctly, and about 75 % of students explain correctly relation (A4.2).

Answering the last question only about 30% of the students did it in the way as above. Many students have tried to answer this question using analytical definition of hyperbola. This is little bit long way, but it is not wrong way.

Analytical definition of hyperbola is that it has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,. \tag{A4.4}$$

in especially chosen (canonical) Cartesian coordinates.

In particular for the hyperbola $H$

$$x^2 - \frac{y^2}{8} = 1\,, \tag{A4.5}$$

On the base of this equation one immediately comes to the condition that the hyperbpla $H$ does not intersect axis $OY$:

$$\text{for } K = (0, t)\,, x^2 - \frac{y^2}{8} = 0 - \frac{t^2}{8} \leq 0 \neq 1\,.$$

This is right conclusion and many students did it. Sure the statements (A4.4) and (A4.5) are correct, but equation (A4.5) holds only in special Cartesian coordinates, not in arbitrary Cartesian coordinates, and one has to prove that the coordinates that we work with are such coordinates. Many students use equation (A4.5) without justifying it, and did not receive the full mark. Only one student who used analytical definition did calculations completely.

**A5.**

*(a) Explain what is meant by the cross-ratio of four collinear points on the projective plane* $\mathbf{RP}^2$*.*

*(b) Four points* $A, B, C, D \in \mathbf{RP}^2$ *are given in homogeneous coordinates by*

$$A = [3 : 3 : 3]\,, \quad B = [14 : 8 : 2]\,, \quad C = [5 : 3 : 1]\,, \quad D = \left[\frac{13}{7} : 1 : \frac{1}{7}\right]\,.$$

*Show that these points are collinear.*

*(c) Calculate their cross-ratio.*

[10 marks]

Students have not special problems answering this question.

This year the question on projective geometry was easy question.

# SECTION B

Answer **TWO** of the THREE questions

**B6.**

*(a) Let* $\mathbf{a} \neq 0$ *be a vector in* $\mathbf{E}^3$.

*Give examples of two orthogonal operators such that the first operator preserves the orientation of* $\mathbf{E}^3$*, the second changes the orientation, and both operators have the eigenvector* $\mathbf{a}$ *with eigenvalue* $-1$*:* $P(\mathbf{a}) = -\mathbf{a}$.

*(b) Show that an arbitrary orthogonal operator* $P$ *in* $\mathbf{E}^3$ *which preserves the orientation and has eigenvector* $\mathbf{a}$ *with eigenvalue* $-1$ *is a rotation operator with respect to an axis which is orthogonal to the vector* $\mathbf{a}$.

[15 marks]

Answering this question you have to understand clear that operator and matrix represented this operator are different objects.

Many students tried to write the orthogonal matrices represented operators, ignoring the fact that these operators have to obey the condition

$$P(\mathbf{a}) = -\mathbf{a}\,. \tag{$B6.1$}$$

and, that vector $\mathbf{a}$ is not necessarily a unit vector.

For example some students have considered matrices

$$P_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{$B6.2$}$$

Of course if you take the orthonormal basis $\mathbf{e}, \mathbf{f}, \mathbf{n}$ such that the first vector is proportional to the vector $\mathbf{a}$, $\mathbf{e} = \frac{\mathbf{a}}{|\mathbf{a}|}$, then the matrices $P_1$, $P_2$ are matrices of linear orthogonal operators which both obey condition (B6.1), such that the $P_1$ preserves orientation, and $P_2$ changes orientation.

Some students have considered the matrices (B6.2) without making any relation of these matrices with orthogonal operators, and with condition (B6.1)

By the way one can notice that the simplest example of orthogonal operator which changes orientation and obeys the condition (B6.1) is the operator $P = -\mathbf{id}$: $P(\mathbf{x}) = -\mathbf{x}$, and it has the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ in an arbitrary basis.

Some students trying to construct examples have used the linear operators

$$P_1(\mathbf{x}) = 2(\mathbf{x}, \mathbf{n})\mathbf{n} - \mathbf{x}\,, \qquad P_2(\mathbf{x}) = -P_1 = \mathbf{x} - 2(\mathbf{x}, \mathbf{n})\mathbf{n}\,, \tag{$B6.3$}$$

which we considered in homeworks. This is right idea, but you have to bother about the fact, that in these examples $\mathbf{n}$ is a unit vector, and it makes these operators orthogonal. If you put $\mathbf{n} = \mathbf{a}$ it will not be orthogonal oprator. Moreover you have to choose the operators such that the vector $\mathbf{a}$ is an eigenvector with eigenvalue $-1$. If you choose $\mathbf{n} = \frac{\mathbf{a}}{|\mathbf{a}|}$, then the operator $P_2$ will be orthogonal operator which does not preserve orientation and condition (B6.1) is obeyed, but for operator $P_1$ it is not obeyed. (You may choose for $P_1$ the vector $\mathbf{n}$ a unit vector which is orthogonal to the vector $\mathbf{a}$.)

One student have considered an "operator" $P(\mathbf{x} = 2(\mathbf{n}, \mathbf{x})\mathbf{x} - \mathbf{x}$ This is not a linear operator at all! (the first term is quadratic over componets of $\mathbf{x}$).

The second part of this question was really difficult question. It has the following 'one line' solution: Let $\mathbf{N}$ be a non-zero vector directed along axis of rotation (it exists due to Euler Theorem) We know that orthogonal operator preserves scalar product ( , ). Hence relations $P(\mathbf{N}) = \mathbf{n}$, $P(\mathbf{a}) = -\mathbf{a}$ imply

$$(\mathbf{a}, \mathbf{N}) = (P(\mathbf{a}), P(\mathbf{N})) = (-\mathbf{a}, \mathbf{N}) = -(\mathbf{a}, \mathbf{N}) \Rightarrow (\mathbf{a}, \mathbf{N}) = 0\,\text{i.e.}\,\mathbf{a} \perp \mathbf{N}\,. \tag{$B6.4$}$$

Nice is not it?

About 10—15 students were trying to solve this problem, but the succesfull attempts were done only by few students. Two students came to the solution which is very close to the solution above. Two students were solving this problem in the following way: Analyzing the operator, they come first to conclusion that if the orthonormla basis is chosen such that first vector of this basis is directed along the vector $\mathbf{a}$, then in this orthonormal basis the operator has the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$

(In other words it is the operator of reflection in the plane which is orthogonal to the vector $\mathbf{a}$). Then if $\mathbf{N}$ is the vector directed along axis, then condition $P(\mathbf{n}) = \mathbf{n}$ means that in the chosen orthonormal basis the first component of the vetor $\mathbf{N}$ vanishes. Hence it is orthogonal to vector $\mathbf{a}$. This way to solution was not easy, especially during exam, when time is limited. One of students who was trying it, did it completely, (second did it almost completely.) My congratulations.

**B7.**

(a) *Formulate the theorem about an integral of an exact* 1*-form over a curve in* $\mathbf{E}^n$ *and prove this theorem.*

(b) *Show that for an arbitrary closed curve* $C$ *in* $\mathbf{E}^3$*,*

$$\int_C y\cos z dx + x\cos z\ dy = \int_C xy \sin z dz\,.$$

The 1-form $\omega = y\cos z dx + x\cos z dy - xy\sin z dz$ is an exact form: $\omega = d(xy\cos z)$, hence

$$\int_C \omega = \int_C y\cos z dx + x\cos z dy - xy\sin z dz = 0 \Rightarrow \int_C y\cos z dx + x\cos z dy = \int_C xy\sin z dz = 0\,.$$

About half of students who were solving this question did it correctly: Typical mistake of another half was that they desperately were trying to prove that 1-forms $\omega_1 = y\cos z dx + x\cos z dy$ and $\omega_2 = xy\sin z dz$ both are exact forms. Of course this is wrong. It is their difference, the form $\omega_1 - \omega_2$ is exact form.

[15 marks]

**B8.**

(a) *Let $C$ be a curve in $\mathbf{E}^3$, defined by the intersection of the conic surface $k^2x^2 + k^2y^2 - z^2 = 0$ with the plane $z + x = 1$. Show that if $|k| > 1$ then this curve is an ellipse.*

(b) *In the case if $k = 2$, choose a parameterisation of this ellipse and calculate the integral of the 1- form $\omega = xdy - ydx$ over this ellipse.*

*To what extent does the answer depend on a choice of parameterisation?*

[15 marks]

If $z = 1 - x$, then

$$k^2x^2 + k^2y^2 - z^2 = k^2x^2 + k^2y^2 - (1-x)^2 = (k^2-1)x^2 + k^2y^2 + 2x - 1 = 0\,,$$

i.e.

$$(k^2-1)\left(x + \frac{1}{k^2-1}\right)^2 + k^2y^2 = \left(1 + \frac{1}{k^2-1}\right) = \frac{k^2}{k^2-1}\,,$$

or

$$\frac{(k^2-1)^2}{k^2}\left(x + \frac{1}{k^2-1}\right)^2 + (k^2-1)y^2 = 1\,. \qquad (B8.1)$$

Equation $(k^2-1)x^2 + k^2y^2 - 2x = 1$ defines on the plane $z = 0$ the orthogonal projection of the curve $C$ (which belongs to the plane $z = 1 - x$) on the plane $OXY$:

$$C\colon \begin{cases}(k^2-1)x^2 + k^2y^2 - 2x = 1\\ z = 1 - x\end{cases}, \text{ and its orthogonal projection } C_{\text{prooj.}}\colon \begin{cases}(k^2-1)x^2 + k^2y^2 - 2x = 1\\ z = 0\end{cases} \qquad (B8.1a)$$

Due to the Theorem (in non-degenerate cases) they define the same conics: $C_{proj}$ is ellipse, parabpla or hyperbola if and only if $C$ is respectively ellipse or parabola or hyperbola.

We see that If $k^2 > 1$, then due to (B8.1) this projection of the curve is an ellipse. Hence the curve is an ellipse also.

b) Now fix $k = 2$. Then our curve is $C$: $\begin{cases} \frac{9}{4}\left(x+\frac{1}{3}\right)^2 + 3y^2 = 1 \\ z = 1 - x \end{cases}$ Choose parameterisation of the ellipse:

$$C: \quad \begin{cases} x = -\frac{1}{3} + \frac{2}{3}\cos t \\ y = \frac{1}{\sqrt{3}}\sin t \\ z = 1 - x = \frac{4}{3} - \frac{2}{3}\cos t \end{cases}, \quad 0 \le t < 2\pi\,. \qquad (B8.2)$$

(in fact it is not necessary to choose the parameterisation of $z(t)$ since differential form does not depend on $dz$)

$$\int_C xdy - ydx = \int (x(t)dy(\mathbf{v}_y(t)) - y(t)dx(\mathbf{v}_x(t)))\,dt.$$

For components of velocity vector we have

$$v_x(t) = \frac{dx(t)}{dt} = -\frac{2}{3}\sin t\,, v_y(t) = \frac{dy(t)}{dt} = \frac{\cos t}{\sqrt{3}}\,, (\mathbf{v} = v_x\partial_x + v_y\partial_y + v_z\partial_z)$$

and $\int_C xdy - ydx =$

$$\int_0^{2\pi}\left(\left(-\frac{1}{3}+\frac{2}{3}\cos t\right)\frac{\cos t}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\sin t\right)\left(-\frac{2\sin t}{3}\right)\right)dt = \frac{\sqrt{3}}{9}\int_0^{2\pi}(2-\cos t)dt = \frac{4\pi\sqrt{3}}{9}\,,$$

($\int_0^{2\pi}\cos t dt = 0$ since the function $\cos t$ is periodical function).

c) Under changing the parameterisation integral does not change (if orientation of new parameterisation is the same), or it changes the sign, if new parameterisation has opposite orientation.

Almost all students who did this exercise, did not focus the attention on the fact that there are two conics in equations (B8.1, B8.1a) , the conic $C$, the intersection of plane with conic surface, and the conic, which is the projection of the conic $C$ on the plane $z = 0$. Due to the Theorem the conic $C$ is the ellipse in the case $|k| > 1$ because its orthogonal projection is the ellipse also.

Respectively the same problem for the second exercise: Many students did not bother about the fact that the form $\omega = xdy - ydx$ has to be integrated over the curve $C$ (see the parameterisation of the curve $C$ in equation (B8,2)) not over the curve $C_{\text{proj}}$. Of course these integrals coincide, since the form $\omega$ does not depend on $z$, but this has to be justified.

I am very happy, that this year almost nobody did mistake, answering on the question about how the answer depends on reparameterisation.

**END OF EXAMINATION PAPER**