Two hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO GEOMETRY

04 June 2018

09:45 – 11:45

Answer **ALL FIVE** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (30 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

EXAM FEEDBACK

SECTION A

Answer **ALL** FIVE questions

A1.

(a) Explain what is meant by saying that two bases in $\mathbf{E}^3$ have opposite orientation.

(b) Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in $\mathbf{E}^3$.

Show that the ordered triples

$$\left\{\frac{4}{5}\mathbf{e} + \frac{3}{5}\mathbf{f}\,,\ \mathbf{g}\,, -\frac{3}{5}\mathbf{e} + \frac{4}{5}\mathbf{f}\right\} \text{ and } \left\{\mathbf{e}\,,\ \frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}\,,\ \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\right\}$$

are both bases, and these bases have opposite orientation.

(c) Are these bases orthonormal bases?

Justify your answer.

[10 marks]

Two bases have the opposite orientation if the transition matrix has negative determiant. Almost all students answered this question right.

Denote by T_1 the transition matrix from the initial orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the first ordered triple $\left\{\frac{4}{5}\mathbf{e} + \frac{3}{5}\mathbf{f}\,,\ \mathbf{g}\,, -\frac{3}{5}\mathbf{e} + \frac{4}{5}\mathbf{f}\right\}$, and denote by T_2 the transition matrix from the initial orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the second ordered triple $\left\{\mathbf{e}\,,\ \frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}\,,\ \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\right\}$. One can see that

$$T_1 = \begin{pmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \end{pmatrix}, \det T_1 = -1\,, \text{and } T_2 = \begin{pmatrix} 1 & \frac{\sqrt{2}}{2} & \frac{2}{3} \\ 0 & \frac{\sqrt{2}}{2} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \det T_2 = \frac{\sqrt{2}}{6}\,.$$

These both triples are bases since matrices T_1 and T_2 are non-degenerate (have non-zero determinants).

Some students showed that the ordered triples are bases, straightforwardly checking that the vectors are linearly independent. This is right way, but it is time consuming, it is easier to check that the determinant of transition matrix is not vanished.

The first basis is orthogonal basis, since it can be checked straightforwardly that all the basic vectors of the first basis have unit length amd they are orthogonal to each other. Equivalently this means that the transition matrix T_1 is orthogonal matrix.

The second basis is not orthonormal basis (in spite of the fact that all the vectors of this basis have unit length) since for example the first and second vectors ae not orthogonal to each other: the scalar product of these vectors is not equal to zero. Equivalently this means that the second matrix is not orthogonal.

Some students came to the right conclusion that the second basis is not orthonormal basis, noting that the orthogonal matrix has to have detrminant ± 1. This is right.

Typical mistakes were;

Some students calculated wrong the determinant of the matrix T_1, and came to the wrong conclusion about not orthonormality of the first basis: if T_1 is orthogonal matrix $\Rightarrow \det T_1 = \pm 1$. Hence the fact that determinant of the transition matrix is not equal to ± 1 implies that the second basis is not orthonormal.

Some students checked that the vectors of the second triple have unit length, and came to the conclusion that this is orthonormal basis. They did not check that the vectors are not orthogonal to each other.

I am very happy that this year only one or two students confuse orthogonal matrices with unimodular matrices: they were coming to the wrong coclusion that $detT_1 = \pm 1$ implies that T_1 is orthogonal matrix. (last years the number of students who confused between unimodular and orthogonal matrices was much more!)

One student concluded that matrix T_1 is orthogonal since

a) its determinant is equal to -1

b) is columns have unit lenght

This is right conclusion, but it is not obvious. The simplest way to justify this conclusion is the following: if edges of parallelepiped have unit length and they are not orthogonal, then its volume (determinant) would be less than 1. (see the one of last etudes in the subsection Geometry in my homepage)

A2.

(a) State the Euler Theorem about rotations.

(b) Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in $\mathbf{E}^3$. Let P be a linear orthogonal operator in $\mathbf{E}^3$ such that it preserves orientation and

$$P(\mathbf{e}) = -\mathbf{e}\,, \quad P(\mathbf{f}) = \mathbf{g}\,.$$

Find $P(\mathbf{g})$. Justify your answer.

(c) Show that P is an operator of rotation with respect to an axis directed along the vector $\mathbf{N} = \mathbf{f} + \mathbf{g}$ on the angle π.

[10 marks]

Students did good this question.

Recall that the Euler Theorem on rotation states:

Let P be an orthogonal operator preserving an orientation of Euclidean space $\mathbf{E}^3$, i.e. operator P preserves the scalar product and orientation. ($(P\mathbf{x}, P\mathbf{y}) = (\mathbf{x}, \mathbf{y})$, $\det P > 0$ (in fact $\det P = 1$).) Then it is a rotation operator with respect to an axis l on the angle φ. Every vector $\mathbf{N}$ directed along the axis is not changed, i.e. the axis is 1-dimensional space of eigenvectors with eigenvalue 1, $P(\mathbf{N}) = \mathbf{N}$. Every vector orthogonal to axis rotates on the angle φ in the plane orthogonal to the axis and $\mathrm{Tr}\, P = 1 + 2\cos\varphi$. (The angle φ is defined up to a sign.)

Formulating Euler Theorem some students did not formulate exactly the relation between axis and eigenvector of the operator P, and did not tell exactly what it means that rotation is on the angle φ (`every vector orthogonal to the axis rotates on the angle` $\varphi intheplaneorthogonaltotheaxis$)

A3.

(a) Give the definition of a differential 1-form on $\mathbf{E}^n$.

(b) Let f be a function on $\mathbf{E}^2$ given by $f(x, y) = x^4 - y^4$.

Let ω be 1-form such that $\omega = df$.

Calculate the value of the 1-form ω on the vector field $\mathbf{A} = x\partial_x + y\partial_y$.

(c) Calculate the integral $\int_C \omega$ of this 1-form over the curve C, which is an upper unit half-circle: $C\colon\ x^2 + y^2 = 1\,, y > 0$.

[10 marks]

The definition of a differential 1-form on $\mathbf{E}^n$ is the following

A differential 1-form ω on $\mathbf{E}^n$ is a function on tangent vectors of $\mathbf{E}^n$, such that it is linear at each point: $\omega(\mathbf{r}, \lambda\mathbf{v}_1 + \mu\mathbf{v}_2) = \lambda\omega(\mathbf{r}, \mathbf{v}_1) + \mu\omega(\mathbf{r}, \mathbf{v}_2)$, $\lambda, \mu \in \mathbf{R}$. Here $\mathbf{v}_1, \mathbf{v}_2$ are vectors tangent to $\mathbf{E}^n$ at the point $\mathbf{r}$, ($\mathbf{v}_1, \mathbf{v}_2 \in T_{\mathbf{r}}\mathbf{E}^n$).

To answer subquestion b) we have:

$\omega = df = d(x^4 - y^4) = 4x^3dx - 4y^3dy$;

$\omega(\mathbf{A}) = (4x^3dx - 4y^3dy)(y\partial_y + x\partial_x) = 4x^3dx(x\partial_x) - 4y^3dy(y\partial_y) = 4x^3dx(\partial_x) - 4y^4dy(\partial_y) = 4x^4 - 4y^4$.

or another solution: $\omega(\mathbf{A}) = df(\mathbf{A}) = \partial_{\mathbf{A}} f = (x\partial_x + y\partial_y)(x^4 - y^4) = 4x^4 - 4y^4$.

and finally to answer the subquestion c) one has to see that $\omega = df$ is an exact 1-form. Since $\omega = df$ is an exact form then

$$\int_C \omega = \int_C df = f(\mathbf{r}_2) - f(\mathbf{r}_1)\,,$$

where $\mathbf{r}_2, \mathbf{r}_1$ are starting and ending points of the upper-half circle $x^2 + y^2 = 1, y > 0$; $\mathbf{r}_{1,2} = (\pm 1, 0)$, and $f(\mathbf{r}_2) = f(\mathbf{r}_1) = (\pm 1)^4 - 0^4 = 1$, i.e. $\int_C \omega = 0$.

Almost all students answered this question, however many students did not notice that ω is an exact form, and calculated straightforwardly the integral $\int_C \omega$. The exactness of form ω facilitates the calculations essentially. Instead calculating the integrand $\omega(\mathbf{v}(t))$ and then calculating the integral, one just do 'one line' calculation:

$$\int_C \omega = \int_C df = f\big|_{\partial C} = f\big|_{-1,0}^{1,0} = f\big|_{-1,0}^{1,0} = 1 - 1 = 0\,.$$

The students who did not use the exactness of the form ω and received the right answer did not loose the mark, but sure they loosed the time.

A4.

(a) Give the definition of an ellipse in $\mathbf{E}^2$ with foci at the given points F_1, F_2.

(b) Let C be an ellipse in the plane $\mathbf{E}^2$ such that it has foci $F_1 = (2,0)$ and $F_2 = (6,0)$, and it passes through the point $(2,3)$.

Show that this ellipse passes through origin.

(c) Find the area of this ellipse.

The full answer on this question is the following

a) Ellipse on the plane $\mathbf{E}^2$ with foci in the given points F_1, F_2 is the locus of the points such that sum of the distances from every point of this locus to the points F_1, F_2 is equal to the given number:

$$C = \{X\colon\ |F_1X| + |F_2X| = a\}\,.$$

(Students who will write only the equation defining the locus C will also recieve the full mark.)

b) Denote origin by $O = (0,0)$. The point $P = (2,3)$ belongs to the ellpise. Hence we have

$$|F_1P| + |F_2P| = 3 + \sqrt{(6-2)^2 + 3^2} = 8\,. |F_1O| + |F_2O| = 2 + 6 = 8 \Rightarrow O \in C\,.$$

c) To calculate area we calculate length of axis of the ellipse. Foci of this ellipse are on the OX axis. The centre of the ellipse is at the point $(4,0)$ since $\frac{2+6}{2} = 4$. Since origin belongs to this ellipse, hence the length of horisontal half-axis is equal to 4.

If the vertical axis has length $2h$, hence the vertices are at the points $(4, \pm h)$. Calculating the sum of distances to foci we have:

$$\sqrt{h^2 + (4-2)^2} + \sqrt{h^2 + (4-6)^2} = 2\sqrt{h^2+4} = 8 \Rightarrow h = \sqrt{12} = 2\sqrt{3}\,.$$

and area is equal to

$$S = \pi \cdot \textit{vertical half-axis} \cdot \textit{horisontal half-axis} = \pi \cdot h \cdot 4 = 8\pi\sqrt{3}\,.$$

Students had no essential problems answering this question.

[10 marks]

A5.

(a) Explain what is meant by the cross-ratio of four collinear points on the projective plane $\mathbf{RP}^2$.

(b) Three points $A, B, C \in \mathbf{RP}^2$ are given in homogeneous coordinates by

$$A = [4:4:2]\,, B = [5:5:1]\,, C = [9:9:3]\,.$$

Show that these points are collinear.

(c) Find a point D on $\mathbf{RP^2}$ such that the cross-ratio $(A, B, C, D) = \frac{1}{2}$.

[10 marks]

The full answer on this question is

a) Let A_1, A_2, A_3, A_4 be four distinct points on a projective plane, and these points belong to the same projective line (they are collinear). Choose an arbitrary affine coordinate u such that it has a finite value at all these points, and it separates the points, i.e. value of this coordinate at different points is different.

Let u_i be the value of this coordinate at the point A_i $(i = 1, 2, 3, 4)$. Then the cross-ratio is equal to

$$(A_1, A_2, A_3, A_4) = \frac{u_1 - u_3}{u_1 - u_4} : \frac{u_2 - u_3}{u_2 - u_4}\,.$$

b) Consider affine coordinates (u, v) of points A, B, C: $u = \frac{x}{z}$, $v = \frac{y}{z}$, where $[x : y : z]$ are homogeneous coordinates (coordinate z of all these points is not equal to zero, and affine coordinates will be finite). In affine coordinates (u, v), $A = (2, 2), B = (5, 5)$ and $C = (3, 3)$. All these points are on the line $v = u$, i.e. they are collinear.

c) Choose an affine coordinate, e.g. $u = \frac{x}{z}$. $u_A = 2$, $u_B = 5$, $u_C = 3$ (this coordinate separates the points A, B, C). Suppose the affine coordinate u takes value $x = u_D$ at the point D. We have $u_A = 2$ $u_B = 5$ $u_C = 3$ $u_D = x$ Calculating cross-ratio we come to

$$\frac{1}{2} = (A, B, C, D) = (2, 5, 3, x) = \frac{(2-3)(5-x)}{(2-x)(5-3)} = \frac{x-5}{2(2-x)} \Rightarrow x = \frac{7}{2}\,.$$

The point D is a finite point. Its coordinate u is equal to $u_D = \frac{7}{2}$. All the points A, B, C, D belong to the same line $v = u$, hence $u_D = v_D = \frac{7}{2}$.

Answering this question some students did not emphasize that cross-ratio is defined only for collinear points.

SECTION B

Answer **TWO** of the THREE questions

B6.

(a) Explain what is meant by saying that a linear operator changes the orientation of vector space.

(b) Let $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ be an orthonormal basis in Euclidean space $\mathbf{E}^3$.
Let P be a linear orthogonal operator in $\mathbf{E}^3$ such that

$$P(\mathbf{N})=\mathbf{N} \text{ for } \mathbf{N}=2\mathbf{e}+\mathbf{f}\,, \quad P(\mathbf{g})=\mathbf{g}\,,$$

and P is not identity operator, $P\neq I$.

Show that the operator P changes the orientation of $\mathbf{E}^3$.

(c) Define a linear operator A such that operator $A\circ P$ is a rotation operator on the angle $\frac{\pi}{2}$ with respect to the axis directed along the vector $\mathbf{N}$.
(To define an operator, you may define its action on the vectors in an arbitrary basis.)

[15 marks]

This was the difficult question, may be the most difficult out of all exam questions. It can be solved in the following way (below we consider also another solutions):

Operator P is identical on the vectors $\mathbf{N}$ and vector $\mathbf{g}$, hence P is identical operator on the plane α spanned by these two vectors. Consider a vector $\mathbf{F}$ which is orthogonal to this plane, e.g. vector $\mathbf{F}=\mathbf{e}-2\mathbf{f}$. Since P is orthogonal operator, then vector $P(\mathbf{F})$ has to be orthogonal to the plane α, i.e. $P(\mathbf{F})=\pm\mathbf{F}$. Since P is not identity operator, then $P(\mathbf{F})=-\mathbf{F}$. We see that P is the reflection operotor with respect to the plane α spanned by the vectors $\mathbf{N}$ and $\mathbf{g}$.

Now we find an operator A such that $A\circ P$ is a rotation operator. It is convenient to choose an orthogonal basis $\mathbf{n}',\mathbf{f}',\mathbf{g}'$ such that $\mathbf{n}'=\frac{\mathbf{N}}{\sqrt{5}}$, $\mathbf{f}'==\frac{\mathbf{F}}{\sqrt{5}}$, and $\mathbf{g}'=\mathbf{g}$. (It is easy to see that this is orthogonal basis.). In this basis $P(\mathbf{n}')=\mathbf{n}', P(\mathbf{f}')=-\mathbf{f}'$ and $P(\mathbf{g}')=\mathbf{g}'$ (The matrix of operator P in this basis is $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$.) Since $A\circ P$ is rotation around axis $\mathbf{n}$ on the angle $\frac{\pi}{2}$ hence we may choose an operator A such that the matrix of operator $A\circ P$ in this basis is $\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$ i.e.

$$A\circ P\colon \{\mathbf{n}',\mathbf{f}',\mathbf{g}'\}\to\{\mathbf{n}',-\mathbf{g}',\mathbf{f}'\}$$

Since $P\colon \{\mathbf{n}',\mathbf{f}',\mathbf{g}'\}\to\{\mathbf{n}',-\mathbf{f}',\mathbf{g}'\}$ hence hence $A\colon \{\mathbf{n}',\mathbf{f}',\mathbf{g}'\}\to\{\mathbf{n}',\mathbf{g}',\mathbf{f}'\}$, i.e.

$$A(\mathbf{n}')=\mathbf{n}'\,,\quad A(\mathbf{f}')=\mathbf{g}'\,,\quad ,A(\mathbf{g}')=\mathbf{f}'\,.$$

About 8-10 students showed that operator P changes the orientation. Some of them did it in the way similar to above[1], and some of them did it using brute force calcualtions (see above) Only few students gave the full solution of this question, giving an example of operator A such that $A \circ P$ is rotation.

Typical mistakes:

a) Many students unexpectably ”’deduced” that condition $P(\mathbf{e}+2\mathbf{f}) = \mathbf{e}+2\mathbf{f}$ implies that

$$P(\mathbf{e}+2\mathbf{f}) = \mathbf{e}+2\mathbf{f} \Rightarrow P(\mathbf{e}) = \frac{1}{2}\mathbf{f}\,, \quad P(\mathbf{f}) = 2\mathbf{e}\,\text{????Why???}$$

b) some students were trying to guess the matrix of the operator in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. However doing it they ignored the fact that that the matrix which tey consider us evidently not orthogonal.

c) some students have choosen the basis $\{\mathbf{N}, \mathbf{F}, \mathbf{g}\}$ where $\mathbf{F} = \mathbf{e} - 2\mathbf{f}$, and this was the right guess (see the solution above). Unfortunately some of them did not realise that these vectors are not unit vectors. This was sourse of confusion.

Some students were trying to calculate straightforrwardly the matrix of operator P in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. It was complicated time consuming problem, but nevertheless this is right way. I am happy to tell that two (or three) students ovecame the calculations and came to the right answer (in the part b) of the question). Describe briefly this solution.

Let operator P has the appearance

$$P = \begin{pmatrix} a & x & e \\ b & y & f \\ c & z & g \end{pmatrix}$$

in the initial orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. Then since $P(2\mathbf{e}+\mathbf{f}) = 2\mathbf{e}+\mathbf{f}$, $P(\mathbf{g}) = \mathbf{g}$ and P is orthogonal operator we come to

$$P = \begin{pmatrix} a & x & 0 \\ b & y & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \quad \text{with} \begin{cases} a^2 + b^2 = 1 \\ x^2 + y^2 = 1 \\ ax + by = 0 \\ 2a + x = 2 \\ 2b + y = 1 \end{cases}$$

We come to TWO solutions:

$$\begin{cases} a = 1 \\ b = 0 \\ x = 0 \\ y = 1 \end{cases}\,, \quad \begin{cases} a = \frac{3}{5} \\ b = \frac{4}{5} \\ x = \frac{4}{5} \\ y = -\frac{3}{5} \end{cases}$$

[1] One student wrote the solution of the fact that the operator P changes orientation in “one line”, he (she) wrote: Operator P is identical on two orthognal vectors on the plane and $P \neq I$, hence it is reflection operator with respect to this plane. Very energetic statement!

The first solution corresponds to identity operator, the second to the operator with $\det P = -1.$, it is

$$P = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These solutions can be derived or by direct calculation which one can essentially simplify noting the fact that the matrix $\begin{pmatrix} a & x \\ b & y \end{pmatrix}$ is orthogonal 2×2 matrix. Not the shortest way to solve the problem. Two-three students using this method found the matrix of operator P as above, but they could not find the operator A.

One student did the following attempt: He considered the matrix of the operator P in the basis $\{\mathbf{N}, \mathbf{f}, \mathbf{g}\}$.

$$P = \begin{pmatrix} 1 & x & 0 \\ 0 & y & 0 \\ 0 & z & 1 \end{pmatrix}$$

Notice that this basis is not orthonormal basis, and matrix of the operator is this basis is not orthogonal matrix. Follow this approach.

Using the fact that P is orthogonal operator we come to equations

$$\begin{cases} (\mathbf{f}, \mathbf{g}) = (P(\mathbf{f}), P(\mathbf{g})) = (x\mathbf{N} + y\mathbf{f} + z\mathbf{g}, g) = z = 0 \\ (\mathbf{f}, \mathbf{N}) = (P(\mathbf{f}), P(\mathbf{N})) = (x\mathbf{N} + y\mathbf{f} + z\mathbf{g}, \mathbf{N}) = 5x + y = 1 \\ (\mathbf{f}, \mathbf{f}) = (P(\mathbf{f}), P(\mathbf{f})) = (x\mathbf{N} + y\mathbf{f} + z\mathbf{g}, x\mathbf{N} + y\mathbf{f} + z\mathbf{g}) = 5x^2 + y^2 + 2xy + z^2 = 1 \end{cases}$$

Solving these simultaneous equations we come to two solutions:

$$\begin{cases} x_1 = 0 \\ y_1 = 1 \\ z_1 = 0 \end{cases}, \quad \begin{cases} x_2 = \frac{2}{5} \\ y_2 = -1 \\ z_2 = 0 \end{cases}$$

The first solutions is the identity operator, but $P \neq I$, hence we come to the second solution: matrix of the operator P in the basis $\{\mathbf{N}.\mathbf{f}, g\}$ is

$$P = \begin{pmatrix} 1 & \frac{2}{5} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \det P = -1$$

To find matrix A in this basis, notice that one can consider operator A such that in the basis $\{\mathbf{N}, \mathbf{f}, \mathbf{g}\}$ $A \circ P = P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Hence one can calculate A in this basis using the fact that

$$A = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = P = \left(\begin{pmatrix} 1 & \frac{2}{5} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

This solution is not the most shortest, but one can go in this way. Unfortunatelly the student who suggested this solution did not continue it in the right way: in particularly he (she) came to the wrong equation, and (what is important) did not notice that the equation has TWO! solutions. On the other hand in this solution you have to distingusih clear between matrices and operators, and this student managed to do it!

B7.

(a) Let C be a curve in $\mathbf{E}^3$, defined by intersection of the conical surface $4x^2 + 4y^2 - 9z^2 = 0$ with the plane $pz = x + 3$, where p is a parameter.
Let $C_{\rm proj}$ be an orthogonal projection of this curve on the plane $z = 0$.

Show that in the case if $p = 3$, then the curve $C_{\rm proj}$ is an ellipse, and the vertex of the conical surface is one of the foci of this ellipse.

(b) Find a value of $p = p_0$ such that the curve $C_{\rm proj}$ becomes a parabola.
Explain why the intersection of the plane $p_0 z = x + 3$, with the plane $z = 0$ is the directrix of this parabola.

[15 marks]

The part a) of the question: putting $p = 3$ in the equation of curve one easy comes to $z = \frac{x}{3} + 1$ and to the following equation for projection of conic

$$C_{\rm proj} : 4x^2 + 4y^2 - 9\left(\frac{x}{3} + 1\right)^2 = 0\,,\,, z = 0$$

After standard transformations we come to

$$3x^2 - 6x + 4y^2 = 9 \Leftrightarrow 3(x-1)^2 + 4y^2 = 12 \Leftrightarrow \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1 \Leftrightarrow \frac{x'^2}{4} + \frac{y'^2}{3} = 1\,,$$

in Cartesian coordinates $\begin{cases} x' = x - 1 \\ y' = y \end{cases}$. This is ellipse with centre at the point $\begin{cases} x' = 0, x = 1 \\ y' = y = 0 \end{cases}$,

with $a = 2$, $b = 3$ and $c = \sqrt{a^2 - b^2} = 1$. Hence foci of this ellipse are at the points $\begin{cases} x' = \pm 1, \\ y' = y = 0 \end{cases}$,

i.e. $\begin{cases} x_1 = 0, x_2 = 2, \\ y_1 = y_2 = 0 \end{cases}$, We see that according the statement of the Theorem, one of the foci is at the vertex of the conical surface.

Many students came to the right answer for the ellipse but about half of them did not succeed to see that vertex of the cone is the focus of this ellipse.

Many students were trying to perform calculations for arbitrary p, and only at the end consider the case $p = 3$. Some of them confused in calculations and came to the wrong equation of the ellipse.

As a consequence they could not see that the vertex of the cone is the focus [2].

Second part of the question was more difficult.

First of all we have to find a value of $p = p_0$ such that conic section becomes parabola.

Many students found this value right: $p_0 = \frac{3}{2}$. in this case the equation of $C_{\rm proj}$ becomes

$$C_{\rm proj} : 4x^2 + 4y^2 - 9\left(\frac{x+3}{\frac{3}{2}}\right)^2 = 0 \ , z = 0 \, ,$$

i.e. in the plane $z = 0$ we have

$$4x^2 + 4y^2 - 4(x+3)^2 = 0 \Leftrightarrow y^2 = 6x + 9 \Leftrightarrow x^2 + y^2 = x^2 + 6x + 9 = (x+3)^2 \, ,$$

i.e. it is locus of the points which is at the equal distance from the origin and the line $x = -3$.

This line is just the intersection of the plane $p_0 z = x + 3$ of conic surface with the plane $z = 0$.

Unfortunately the essential part of students who calculated right the value p_0 did mistakes in the further calculations, and could not find the focus and directrix of the parabola. About 4-5 students did fully the second part of the question using the considerations above or using the standard formulae for parabola. One may come to the answer using polar coordinates, but unfortunately nobody did it in this way.

B8.

(a) Explain what is meant by a projective transformation of the projective line **RP**.

(b) Let A_1, A_2, A_3, A_4 be four points on the projective line **RP** and u be an affine coordinate on **RP** such that,

$$u_{A_1} = 1 \, , \quad u_{A_2} = 2 \, , \quad u_{A_3} = 3 \, , \quad u_{A_4} = 4 \, .$$

Give an example of projective transfromation of the projective line, which transforms the point A_4 to infinity.

(c) Suppose that A, B, P, Q are four points on **RP** such that the affine coordinate u of these points obeys the relation

$$\frac{u_A - u_P}{u_A - u_Q} = -\frac{u_B - u_P}{u_B - u_Q} \, .$$

(We assume that all these points have finite affine coordinate u.)
Let F be a projective transformation of the projective line, such that point Q transforms to

[2] The fact that vertex $x = y = 0$ is the focus of the ellipse can be deduced in polar coordinates also: to see that point $x = y = 0$ is a focus of this ellipse it is useful to use polar coordinates. In polar coordinates $x = r\cos\varphi, y = r\sin\varphi, z = h$, equation of the curve C is $C\colon \begin{cases} 4r^2 = 9h^2 \\ ph = r\cos\varphi + 3 \end{cases}$, i.e. $C\colon \begin{cases} 2r = \pm 3h \\ ph = r\cos\varphi + 3 \end{cases}$. If $p = 3$ then $C\colon \begin{cases} 2r = 3h \\ 3h = r\cos\varphi + 3 \end{cases}$ since $h > 0$, i.e. $C_{\rm proj}\colon\ 2r = r\cos\varphi + 3$, i.e. $r = \frac{3}{2-\cos\varphi}$. This is equation of ellipse in polar coordinates with focus at origin. Unfortunately nobody did it in polar coordinates.

infinity.
Show that this projective transformation transforms the point P to the midpoint of the transform of the line segment $[AB]$, i.e.

$$u_{P'} = \frac{u_{A'} + u_{B'}}{2} \,,$$

where $u_{A'}$, $u_{B'}$, and $u_{P'}$ are affine coordinates of points A, B, P after the projective transformation F.

[15 marks]

Just two comments on this question:

1) This question was not very difficult, but may be because this chapter was the last one, many students have chosen not to try solve this problem (students have to solve two of three last problems.) On the other hand I am happy to tell that the percentage of students who attempted to solve this problem and solved it was high, much higher than for the questions B6 and B7: (almost all of students who attempted this question recieved $\geq 6/15$).

2) this year projective transformations will not be included in examination.

Projective transformation of $\mathbf{RP}$ in homogeneous coordinates is a transformation which transforms any point $[x : y] \in \mathbf{RP}$ to the point

$$[x', y'] = [ax + by : cx + dy] \,, \tag{B7.1}$$

where a, b, c, d are parameters such that matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-degenerate:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0 \,. \tag{B7.2}$$

Many students who were explaining what is it a projective transformation, have forgotten to emphasize this condition.

Condition that the point $[4 : 1]$ transforms to infinity, means that

$$cx_4 + dy_4 = 4c + d = 0 \,. \tag{B7.3}$$

Choose just for example the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -4 \end{pmatrix}$. This is invertible matrix: we see that $A_4 = [4 : 1]$ transforms to $[x' : y'] = [4 : 0] = \infty$. One can write this transformation in affine coordinate:

$$u' = \frac{x'}{y'} = \frac{x}{x - 4y} \frac{\frac{x}{y}}{\frac{x}{y} - 4} = \frac{u}{u - 4}$$

Of course one may consider other examples: any transformation (B7.1) obeying conditions (B7.2) and (B7.3) is the projective transformation which sends the point A_4 to the infinity.

The last subquestion, subquestion c) was the hardest in this question.

One can see that condition

$$\frac{u_A - u_P}{u_A - u_Q} = -\frac{u_B - u_P}{u_B - u_Q} \tag{B7.4}$$

means that coross ratio of these points is equal to -1:

$$\frac{u_A - u_P}{u_A - u_Q} = -\frac{u_B - u_P}{u_B - u_Q} \Leftrightarrow \frac{(u_A - u_P)(u_B - u_Q)}{(u_A - u_Q)(u_B - u_P)} = (A, B, P, Q) = -1\,,$$

i.e. cross ratio of this points is equal to -1 (points are harmonic conjugate).

Projective transformation preserves cross-ratio!

Denote by $A' = F(A)$ $B' = F(B)$ $P' = F(P)$ $Q' = F(Q)$ We have:

$$(A, B, P, Q) = -1 = (A', B', P', Q') = \frac{(u_{A'} - u_{P'})(u_{B'} - u_{Q'})}{(u_{A'} - u_{Q'})(u_{B'} - u_{P'})} =$$

$$\frac{(u_{A'} - u_{P'})(u_{B'} - \infty)}{(u_{A'} - \infty)(u_{B'} - u_{P'})} = \frac{u_{A'} - u_{P'}}{u_{B'} - u_{P'}} = -1 \Rightarrow u_{A'} - u_{P'} = u_{P'} - u_{B'} \Rightarrow u_{P'} = \frac{u_{A'} + u_{B'}}{2}\,.$$

The key point of this solution is that projective transformation preserves the cross-ratio.

Some students noted that relation (B7.4) between coordinates of the points A, B, C and D implies that

$$u_P = \frac{\frac{2u_A u_B}{u_Q} - u_A - u_B}{\frac{u_A + u_B}{u_Q} - 2} \tag{B7.5}$$

(this can be checked by elementary transformations) Under projective transformation F this expression becomes

$$u_{P'} = \frac{\frac{2u_{A'} u_{B'}}{u_{Q'}} - u_{A'} - u_{B'}}{\frac{u_{A'} + u_{B'}}{u_Q} - 2}\,, \tag{B7.6}$$

and the right hand side of the last expression is obviously equal to $\frac{-u_{A'} - u_{B'}}{-2}$ since $u_{Q'} = \infty$.

Hence

$$u_{P'} = \frac{-u_{A'} - u_{B'}}{-2} = \frac{u_{A'} + u_{B'}}{2}\,.$$

This is elegant solution. It is based on the fact that equation (B7.5) is an invariant: under projective transformation F equation (B7.5) transforms to equation (B7.6) since they both are equivalent to the condition that cross-ratio is equal to -1. Unfortunately almost all students who suggested this solution did not explain why (B7.5) implies (B7.6) and they received partial mark for this question.

Finally I would like to tell that any student who would like to discuss the questions above is welcomed to contact with me.