**Two hours**

**THE UNIVERSITY OF MANCHESTER**

INTRODUCTION TO GEOMETRY

05 June 2019

X.XX – X.XX

Answer **ALL FIVE** questions in Section A (50 marks in total)
Answer **TWO** of the THREE questions in Section B (30 marks in total)
If more than TWO questions in Section B are attempted,
the credit will be given for the best TWO answers.

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The use of electronic calculators is not permitted.

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**FEEDBACK**

# SECTION A

Answer **ALL** *FIVE* questions

**A1.**

*(a) Explain what is meant by saying that two bases in $\mathbf{E}^3$ have the same orientation.*

*(b) Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in $\mathbf{E}^3$. Consider the ordered triple $\{\mathbf{e}+\mathbf{f}, \mathbf{e}+2\mathbf{f}, \mathbf{e}+\mathbf{f}+\mathbf{g}\}$. Show that this triple is a basis. Show that the bases $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and $\{\mathbf{e}+\mathbf{f}, \mathbf{e}+2\mathbf{f}, \mathbf{e}+\mathbf{f}+\mathbf{g}\}$ have the same orientation.*

*(c) Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be two bases in $\mathbf{E}^3$ which have the opposite orientations. Consider in $\mathbf{E}^3$ the ordered triples $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\lambda\mathbf{f}, \mathbf{g}\}$, where $\lambda$ is a coefficient, $\lambda \in \mathbf{R}$. Find all values of $\lambda$ such that the ordered triple $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\lambda\mathbf{f}, \mathbf{g}\}$ is a basis and this basis and the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientations.*

[10 marks]

Almost all students answered this question. Just two remarks:

- Answering the subquestion (b) you consider the transition matrix $T$ between initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and the ordered triple $\{\mathbf{e}+\mathbf{f}, \mathbf{e}+2\mathbf{f}, \mathbf{f}+\mathbf{g}\}$. and $\det T = 1$. Hence the triple is basis since the transition matrix is not degenerate, and this basis and initial basis have the same pparameterisation since $\det T > 0$. Justifying the fact the triple is a basis some students instead just using that the transition matrix is non-degenerate , i.e. $\det T \neq 0$, straightforwardly have checked the linear independence of basic vectors of the new basis. Sure this is not a mistake, but this is time consuming.
- Answering the last subquestion:we have to consider transition matrix $T$ between initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and the ordered triple $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\lambda\mathbf{f}, \mathbf{g}\}$. One can see that $\det T = 1-\lambda$. This triple is a basis if and only if $\det T = 1-\lambda \neq 0$. Since the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have opposite orientation, hence the new basis $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\lambda\mathbf{f}, \mathbf{g}\}$ and the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ have the same orientation: $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\lambda\mathbf{f}, \mathbf{g}\} \sim \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ since there are exactly two equivalence classes of orientation, i.e. the determinant of transition matrix has to be positive: $\det T = 1-\lambda > 0$, and $\lambda < 1$.

  Almost all students answered right this subquestion, however some students did not provide justification of the answer.

**A2.**

*(a) State the Euler Theorem about rotations.*

*(b) Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in Euclidean vector space $\mathbf{E}^3$, and let $P$ be a linear operator acting on this space such that.*

$$P(\mathbf{e}) = \mathbf{f}\,, \quad P(\mathbf{f}) = \mathbf{g}\,, \quad P(\mathbf{g}) = \mathbf{e}\,.$$

*Show that $P$ is a rotation operator.*

(c) *Find the axis of rotation defined by the operator $P$.*

[10 marks]

Almost all students answered this question. Just the following remarks:

On subquestion subquestion a):
The Euler Theorem on rotation states:
*Let $P$ be an orthogonal operator preserving an orientation of Euclidean space $\mathbf{E}^3$, i.e. operator $P$ preserves the scalar product and orientation. ($(P\mathbf{x}, P\mathbf{y}) = (\mathbf{x}, \mathbf{y})$, $\det P > 0$ (in fact $\det P = 1$).) Then it is a rotation operator with respect to an axis $l$ on the angle $\varphi$. Every vector $\mathbf{N}$ directed along the axis is not changed, i.e. the axis is $1$-dimensional space of eigenvectors with eigenvalue $1$, $P(\mathbf{N}) = \mathbf{N}$. Every vector orthogonal to axis rotates on the angle $\varphi$ in the plane orthogonal to the axis and $\mathrm{Tr}\, P = 1+2\cos\varphi$. (The angle $\varphi$ is defined up to a sign.)*

Formulating of Euler Theorem some students gave non-complete formulation. (the role of axis, the rotation of the vectors in orthogonal plane).

Two or three students claim the Theorem for arbitrary $n$-dimensional Euclidean space. In fact Euler theorem is wrong for $n > 3$.

Answering subquestion b) some students wrote:

`operator` $P$ `is orthogonal since` $\det P = 1$ *This is wrong, and this mistake is very bad !*

Many students answering the subquestion c) (the axis is directed along the vector $\mathbf{N} = \mathbf{e} + \mathbf{f} + \mathbf{g}$) did not justify why the linear operator $P$ is a rotation operator: you have to show that it is rotation operator, i.e. it is orthogonal operator in $\mathbf{E}^3$ which preserves orientation.

**A3.**

(a) *Give a definition of a differential $1$-form in $\mathbf{E}^n$.*

(b) *Consider in $\mathbf{E}^2$ the differential $1$-form $\omega = xdy - ydx$ and the circle $C$ defined by the equation $x^2 + y^2 = 2x$.*
*Choose a parameterisation of this circle and calculate the integral $\int_C \omega$.*

(c) *How does the answer depend on a choice of parameterisation?*

[10 marks]

Almost nobody had problems to give a definition of $1$-form.
Surprisingly many students had problems when calculating $\int_C \omega$. Look at calculations:
Note that a curve $C\colon x^2 + y^2 = 2x$ is a circle: $(x-1)^2 + y^2 = 1$ with centre at the point $(1, 0)$ and radius $1$. Choose a parameterisation of this circle e.g.

$$\begin{cases} x = 1 + \cos t \\ y = \sin t \end{cases} \quad , \quad 0 \leq t \leq 2\pi\,. \tag{A3a}$$

We have that the velocity vector of curve $\mathbf{v} = -\sin t\partial_x + \cos t\partial_y$, then calculate the value of differential form $\omega$ on the velocity vector:

$$\omega(\mathbf{v}) = (xdy - ydx)\left(-\sin t\partial_x + \cos t\partial_y\right) = x\cos t + y\sin t = \tag{A3b}$$

$$(1+\cos t)\cos t+\sin^2 t = 1+\cos t \qquad (A3c)$$

Then $\int_C \omega = \int_0^{2\pi} \omega(\mathbf{v}(t))dt = \int_0^{2\pi}(1+\cos t)dt = 2\pi$.

If we choose another parameterisation, then the answer will be the same $\int_C \omega = 2\pi$ if this new parameterisation and the parameterisation (A3a) have the same orientation, and the answer will be $\int_C \omega = -2\pi$ if this new parameterisation and the parameterisation (A3a) have the opposite orientations,

Many students did not calculate explicitly the value of $1$-form on velocity vector (equation (A3b)). About 5-8 students came to conclusion wrong conclusion in equation (A3c) that $\omega(\mathbf{v}) = \cos t$. This implies wrong answer that $\int \omega$ is equatl to zero, and it does not depend on orientation of the curve (since $0 = -0$.)

**A4.**

*(a) Give the definition of a hyperbola with foci at the given points $F_1, F_2$.*

*(b) Write down the equation defining a hyperbola in Cartesian coordinates $(x, y)$ such that it passes through the point $(2, 0)$, and foci of this hyperbola are at the points $F_{1,2} = (\pm 2\sqrt{2}, 0)$.*

*(c) Show that the line $y = x$ does not intersect this hyperbola.*

[10 marks]

Subquestion a) No problem

Subquestion b) This hyperbola possesses foci at the $OX$ axis at the points symmetric with respect to origin, hence it is defined by the canonical equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,.$$

This hyperbola interesects $OX$ axis at the points $(\pm a, 0)$ and has foci at the points $(\pm\sqrt{a^2+b^2}, 0)$. We have $a = 2$ and $\sqrt{a^2+b^2} = 2\sqrt{2}$, thus $a = b = 2$, and equation of the hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{4} = 1\,. \qquad (A4a)$$

Many students were confused in these calculations.

Subquestion (c). Answer immediately follows from equation (A4a) for the hyperbola: At every point $(x, y)$ of the hyperbola, $y^2 = x^2 - 4 < x^2$, $y \neq x$.

Solving c) some students were trying to prove that $y = x$ does not belong to the hyperbola showing that the difference of the distances for these points is not equal to $4$. Of course it is not the shortest way to come to the answer.

One student used these considerations and he (she) suggested the following solution of subquestion c): In the case if line $y = x$ intersects with hyperbola, then there exists $x$ such that

$$|d\left((x,x),F_1\right) - d\left((x,x),F_2\right)| = \left|\sqrt{(x+2\sqrt{2})^2+x^2} - \sqrt{(x-2\sqrt{2})^2+x^2}\right| = 4 \qquad (A4b)$$

Taking squares of left and right hand sides we come to the equivalent equation

$$4x^2+16-2\sqrt{(x+2\sqrt{2})^2+x^2}\sqrt{(x-2\sqrt{2})^2+x^2} = 4x^2+16-2\sqrt{64+4x^4} = 16\,. \qquad (A4c$$

This equation evidently has no solutions (the line $y = x$ does not intersect the hyperbola) since $2\sqrt{64+4x^4} > 4x^2$. This is beautiful, in spite of the fact that it this is not the quickest solution.

Finally one remark

**Remark** Geometrical meaning of this consideration is that the line $y = x$ is one of asymptots of the hyperbola $x^2 - y^2 = 4$;hyperbola tends to this line, but it never intersects with this line. Indeed it follows from equation (A4b) and (A4c) that

$$|d((x,x),F_1) - d((x,x),F_2)| = \sqrt{4x^2 + 16 - 2\sqrt{64+4x^4}}\,.$$

It is easy to see that this expression is less than 4 and it tends to 4 if $x \to \pm\infty$: the difference of the distances tends to zero, and it never becomes zero:

$$|d((x,x),F_1) - d((x,x),F_2)| = \left|\sqrt{(x+2\sqrt{2})^2 + x^2} - \sqrt{(x+2\sqrt{2})^2 + x^2}\right| =$$

$$\left|\sqrt{2x^2 + 4\sqrt{2}x + 8} - \sqrt{2x^2 + 4\sqrt{2}x + 8}\right| = \frac{8\sqrt{2}|x|}{\sqrt{2x^2+4\sqrt{2}x+8} + \sqrt{2x^2+4\sqrt{2}x+8}} =$$

$$\frac{8\sqrt{2}|x|}{\sqrt{2}|x|\sqrt{1+\frac{2\sqrt{2}}{x}+\frac{8}{2x^2}} + \sqrt{2}x\sqrt{1-\frac{2\sqrt{2}}{x}+\frac{8}{2x^2}}} = \frac{8}{\sqrt{1+\frac{2\sqrt{2}}{x}+\frac{8}{2x^2}} + \sqrt{1-\frac{2\sqrt{2}}{x}+\frac{8}{2x^2}}}$$

Sure these considerations are extracurricula.

## A5.

*(a) Two points $A, B$ on the projective plane $\mathbf{RP}^2$ are given in homogeneous coordinates*

$$A = [2:4:2]\,,\quad B = [2:4:1]\,.$$

*Find the point $C = [x:y:z]$ on $\mathbf{RP}^2$ such that $y = z$ and the three points $A, B, C$ are collinear.*

*(b) On the projective line passing through the three collinear points $A$, $B$ and $C$ find a point $D$ which is harmonic conjugate to the points $A, B$ and $C$, i.e. the cross-ratio $(A, B, C, D)$, of these points is equal to $-1$.*

*(c) Find the cross-ratio $(A, C, B, D)$.*

[10 marks]

Almost all students have no problems to answer subquestions a) and b). Answering subquestion c) students did not need to caclulate explicitly the cross-ratio $(A, C, B, D)$. They just may come to the right answer noting that

$$(A, C, B, D) = 1 - (A, B, C, D) = 1 - (-1) = 2$$

Of course you could come to the same answer making the straightforward calculations:

$$(A, C, B, D) = \frac{(u_A - u_B)(u_C - u_D)}{(u_A - u_D)(u_C - u_B)} = \frac{(1-2)\left(\frac{1}{2} - \frac{5}{4}\right)}{\left(1 - \frac{5}{4}\right)\left(\frac{1}{2} - 2\right)} = 2\,.$$

However this straightforward solutions is time consuming, and many students came to the wrong answer doing these straightforward calculations. Students who did not make mistakes during these straightforward calculations received the full credits for the answer.

## SECTION B

Answer **TWO** of the three questions

### B6.

(a) *Let $P$ be a linear orthogonal operator in Euclidean space $\mathbf{E}^3$ such that this operator obeys equation*

$$P^6 = \mathbf{id}\,. \tag{1}$$

*Suppose that the operator $P$ preserves orientation of $\mathbf{E}^3$.*
*Find all possible values of the trace of the operator $P$.*

(b) *Give an example of a linear orthogonal operator $P$ which obeys equation* (1)*, but does not preserve orientation of $\mathbf{E}^3$.*

[15 marks]

This was not an easy question, and I am happy that about 20-25 students were trying succesfully to solve this problem: they have guessed that

$$\mathrm{TrP} = 1 + 2\cos\varphi\,,\ \text{where } \varphi \text{ is an angle such that } 6\varphi = 2\pi k, k = 0, \pm 1, \pm 2, \ldots \tag{B6a}$$

The curreful analysis of this relation leads to the fact that

$$\begin{cases} \varphi = 0 & \mathrm{Tr}\, P = 1 + 2\cos\varphi = 1 + 2\cos 0 = 3 \\ \varphi = \frac{2\pi}{6} & \mathrm{Tr}\, P = 1 + 2\cos\varphi = 1 + 2\cos\frac{2\pi}{3} = 2 \\ \varphi = \frac{4\pi}{6} & \mathrm{Tr}\, P = 1 + 2\cos\varphi = 1 + 2\cos\frac{2\pi}{3} = 0 \\ \varphi = \frac{6\pi}{6} & \mathrm{Tr}\, P = 1 + 2\cos\pi = 1 + 2\cos\pi = -1 \end{cases}$$

The subquestion b) was not difficult, and it was solved by almost all students who have guessed e1quation (B6a). The simplest example of such an operator is the operator $P = -\mathbf{id}$

About 7 -10 students were trying to consider the operator in $\mathbf{E}^2$ instead $\mathbf{E}^3$.

### B7.

(a) *State the theorem about an integral of an exact $1$-form over a curve in $\mathbf{E}^n$ and prove this theorem.*

(b) *Explain why $1$-form $\omega = dy + ydx$ is not an exact form.*
*Give an example of a function $f = f(x, y)$ such that $f \not\equiv 0$ and the $1$-form $\sigma = f\omega$ is an exact form. Justify your answer.*

[15 marks]

The first subquestion is not very easy but it is bookwork question.

About half of students who answered this subquestion wrote the complete formulation of the Theorem, but they did not write the proof.

Now about the subquestion b) Many students succesfully explained why the form $\omega = dy + ydx$ is not an exact form.

Then students had to give an example of a function $f$ such that $f\omega$ is an exact form. About 20 students gave the following example: $f = e^x$. Yes, $f\omega$ is an exact form:

$$f\omega = e^x(dy + ydx) = d(e^x y)\,.$$

About 10 students gave another example : function $f = \frac{1}{y}$:

$$f\omega = \frac{dy + ydx}{y} = d(x \log y)$$

Yes, $f\omega$ is exact also, but in this case the 1-form is not defined at $y = 0$.

**Remark** (this is out of curriculum) You may wonder how look the general example. Solve the corresponding differential equation:

$$f(dy + xdy) = d\Phi(x, y) \Rightarrow \Phi(x, y) = F(ye^x) \Rightarrow f = e^x F'\,,$$

where $F = F(x)$ is an arbitrary function on one variable.

One can prove that for an arbitrary 1-form $\omega = a(x, y)dx + b(x, y)dy$ in $\mathbf{E}^2$ there is a function $f$ such that $f\omega$ is an exact form. This property has very powerful applications, e.g. using it one may define temperature in terms of the first law of thermodynamics[1]

Some students were trying to find a function $f$ which is polynomial function. It was in vain: the function $f$ has to have the following expression: $f = e^x G(ye^x)$ (see the remark above)

## B8.

(a) *Let $C$ be a conic section which is the intersection of the conical surface $x^2 + y^2 = z^2$ with the plane $z = \frac{1}{2}x + 1$.*
*Let $C_{\rm proj}$ be an orthogonal projection on the plane $z = 0$ of the conic section $C$.*
*Show that the curve $C_{\rm proj}$ is an ellipse and find foci of this ellipse.*

(b) *Calculate the integral of 1- form $\omega = \frac{xdy - ydx}{x^2 + y^2}$ over the ellipse $C_{\rm proj}$.*

[15 marks]

The first subquestion was not easy but it was discussed during homeworks.

Conic section $C_{\rm proj}$ can be defined as set of points $(x, y)$ on the plane $\mathbf{E}^2$ such that $z = \sqrt{x^2 + y^2} = \frac{1}{2}x + 1$. We have that

$$\sqrt{x^2 + y^2} = \frac{1}{2}x + 1 \Rightarrow x^2 + y^2 = \frac{1}{4}x^2 + x + 1 \Leftrightarrow \frac{9}{16}\left(x - \frac{2}{3}\right)^2 + \frac{3}{4}y^2 = 1\,.$$

The last equation is the equation of the ellipse We proved that $C_{\rm proj}$ is an ellipse. We know that foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ $(a > b$ ) are at the points $F_{1,2} = (\pm\sqrt{a^2 - b^2}, 0)$. Hence for the ellipse $C_{\rm proj}$ its foci are at the points

$$y_{1,2} = 0\,, \quad x_{1,2} - \frac{2}{3} = \pm\sqrt{\frac{16}{9} - \frac{4}{9}} = \pm\frac{2}{3}$$

thus $F_1 = (0, 0)$, $F_2 = \left(\frac{4}{3}, 0\right)$. (These calculations justify that one of the foci is at the vertex of the cone.)

[1] $\delta Q = dU + pdV \Rightarrow T\colon\ \frac{1}{T}\delta Q = \frac{1}{T}(dU + pdV) = dS$, where $S$ is an entropy.

Many students who were solivng this question did mistake in calculations. Student who firmly understands that one of the foci **must** be in the vertex of the cone may avoid these mistakes.

The last subquestion turned to be very difficult. In fact the form $\omega = d\varphi$. Few students noticed it, they concluded that the form is exact, hence the integral over closed form vanishes. This is wrong: the form $\omega = d\varphi$ is not exact since the function $\varphi$ is not well-defined on $\mathbf{E}^2$. The integral of the form $\omega = d\varphi$ over closed curve is equal to $2\pi$ if an origin belongs to the interior of this curve. In other case it is equal to zero.

Only one student solved this problem.

**END OF EXAMINATION PAPER**