Homework 0

In this Homework we will just recall some necessary background material from linear algebra

1 Consider sets

$$V = \{x: \quad ax^2 + bx + c\,, \quad a,b,c,x \in \mathbf{R}\}\,, \qquad T = \{x: \quad x^2 + px + q\,, \quad p,q,x \in \mathbf{R}\}\,,$$

a) Explain why a set V is a vector space, and a set T is not a vector space (with respect to natural operations of mulitplication and addition of polynomials)

b) Explain why polynomials $1, x, x^2$ are linearly independent in V.

c) Calculate dimension of V.

2 Show that the vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ in vector space V are linearly dependent if at least one of these vectors is equal to zero.

3 Show that arbitrary three vectors in $\mathbf{R}^2$ are linearly dependent.

Consider the following vectors in $\mathbf{R}^2$

$$\mathbf{e}_1 = (1,0)\,, \qquad \mathbf{e}_2 = (0,1)\,, \qquad \mathbf{a} = (2,3)\,, \qquad \mathbf{b} = (3,0)\,, \tag{1}$$

Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis in $\mathbf{R}^2$.

Show that $\{\mathbf{a}, \mathbf{b}\}$ is a basis in $\mathbf{R}^2$.

Show that $\{\mathbf{e}_1, \mathbf{b}\}$ is not a basis in $\mathbf{R}^2$.

4 a) Show that $\langle \mathbf{x}, \mathbf{y} \rangle = x^1y^1 + x^2y^2$ *does not define a scalar product* in $\mathbf{R}^3$.

b) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ defines a scalar product in $\mathbf{R}^3$.

c) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not define a scalar product* in $\mathbf{R}^3$.

f†) Find necessary and sufficient conditions for entries a, b, c of symmetrical matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that the formula

$$(\mathbf{x}, \mathbf{y}) = (\,x^1, x^2\,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

defines a scalar product in $\mathbf{R}^2$.

5 Let $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ be three vectors in 3-dimensional Euclidean space $\mathbf{E}^3$ such that all these vectors have unit length and they are pairwise orthogonal.

Show explicitly that the ordered set of these vectors $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is a basis.

6 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors in 3-dimensional Euclidean space $\mathbf{E}^3$ such that vectors $\mathbf{a}$ and $\mathbf{b}$ have unit length, and are orthogonal to each other and vector $\mathbf{c}$ has length $\sqrt{3}$ and it forms an angle $\varphi = \arccos \frac{1}{\sqrt{3}}$ with vectors $\mathbf{a}$ and $\mathbf{b}$.

Show that the ordered set $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ of vectors is an orthonormal basis in $\mathbf{E}^3$.

7 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^3$. Consider the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ which is expressed via basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the following way:

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where λ is an arbitrary coefficient)?

i) Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis in $\mathbf{E}^3$. Is this basis an orthonormal basis of $\mathbf{E}^3$?

ii) Write down explicitly transition matrix which transforms the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered set of the vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. Is this matrix non-degenerate, or no? Is this matrix orthogonal?

Answer question i) using the properties of corresponding transition matrices.

(you have to consider all cases a),b) c) and d)).

8† Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}),$$

where $\mathbf{x}, \mathbf{y}$ are arbitrary two vectors and $(\ ,\ )$ is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ *consider the quadratic polynomial* $At^2 + 2Bt + C$ *where* $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})$. *Show that this polynomial has at most one real root and consider its discriminant.*