Homework 1

1 Let $\mathbf{R}^2$ be an affine space of points. Consider in $\mathbf{R}^2$ points $A = (2,3)$ and $B = (6,6)$.

a) Find the length of the segment AB

b) Find a point C in $\mathbf{R}^2$ such that vector AC has unit length and it is orthogonal to the vector $\vec{AB}$

2 In affine space $\mathbf{R}^2$ consider points $A = (2,1)$, $B = (2+a, 1+b)$ and $C = (2+p, 1+q)$, where a, b, p, q, r are arbitrary parameters.

Calculate the area of the triangle $\triangle ABC$ and compare the answer with determinant of the matrix $\begin{pmatrix} a & b \\ p & q \end{pmatrix}$.

3 Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in 3-dimensional Euclidean space $\mathbf{E}^3$.

Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be the row of three vectors in this space, and let A be the transition matrix from the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the row $\{\mathbf{a}, \mathbf{b}, c\}$:

$$\{\mathbf{a}, \mathbf{b}, c\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\} A\,.$$

Consider the cases

$$\text{a) } A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 7 \end{pmatrix}, \qquad \text{b) } A = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -\sqrt{3} \end{pmatrix}, \qquad \text{c) } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Show that in the case a) the row $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis. Show that this basis is not an orthonormal basis.

Show that in the case b) the row $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis, and this is the orthonormal basis. Find in this case the inverse transition matrix, the transition matrix from the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ to the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Show that in the case c) the row $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is not a basis.

4 Let P be a linear operator in 2-dimensional vector space V. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis in V such that

$$P(\mathbf{e}_1) = 7\mathbf{e}_1 + 9\mathbf{e}_2\,, P(\mathbf{e}_2) = 2\mathbf{e}_1 + 3\mathbf{e}_2\,.$$

Consider in V the new bases $\{\mathbf{f}_1, \mathbf{f}_2\}$ and $\mathbf{g}_1, \mathbf{g}_2$ such that

$$\mathbf{f}_1 = \frac{1}{2}\mathbf{e}_1\,, \quad \mathbf{f}_2 = 3\mathbf{e}_2 \qquad \text{and} \quad \mathbf{g}_1 = \mathbf{e}_1 + \mathbf{e}_2\,, \quad \mathbf{g}_2 = \mathbf{e}_2\,,$$

($\{\mathbf{f}_1, \mathbf{f}_2\}$ and $\mathbf{g}_1, \mathbf{g}_2$ are bases since evidently vectors $\mathbf{f}_1, \mathbf{f}_2$ are linearly independent, and vectors $\mathbf{g}_1, \mathbf{g}_2$ are also linearly independent.)

Write down the matrices of the operator P in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, and in the new bases $\{\mathbf{f}_1, \mathbf{f}_2\}$ and $\{\mathbf{g}_1, \mathbf{g}_2\}$.

5 Let A be a linear operator in 2-dimensional vector space V such that for a given basis $\{\mathbf{e}, \mathbf{f}\}$,

$$A(\mathbf{e}) = 27\mathbf{e} + 40\mathbf{f}\,, A(\mathbf{f}) = -16\mathbf{e} - \frac{71}{3}\mathbf{f}\,.$$

Write down the matrix of the operator A in this basis.

Consider the pair of vectors $\{\mathbf{e}', \mathbf{f}'\}$ such that $\mathbf{e}' = 2\mathbf{e} + 3\mathbf{f}$ and $\mathbf{f}' = 3\mathbf{e} + 5\mathbf{f}$.

Show that an ordered set of vectors $\{\mathbf{e}', \mathbf{f}'\}$ is also a basis, and find a matrix of the operator A in the new basis.

Do vectors $\mathbf{e}', \mathbf{f}'$ are eigenvectors of the linear operator A?

Calculate the determinant and trace of operator A.

6† Let V be a space of functions, which are solutions of differential equation

$$\frac{d^2y(x)}{dx^2} + p\frac{dy(x)}{dx} + qy(x) = 0\,, \tag{1}$$

where parameters p, q are equal to $p = -7$ and $q = 12$.

Show that V is 2-dimensional vector space.

Find a basis in this vector space, and write down the operator A in this basis.

Show that the differentiation $A = \frac{d}{dx}$ is a linear operator on the space V which transforms every vector from V to another vector on V.

Find determinant and trace of this linear operator.