Homework 2

0 Let P be an operator in 2-dimensional vector space V such that in the given basis $\{\mathbf{e}_1\,, \mathbf{e}_2\}$ the matrix of this operator is

$$\begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}.$$

a) write down the action of the operator P on vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

b) Show without 'long' calculations that matrix of this operator in the 'suitable' basis is $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. *(You may use without a proof that this operator indeed has two linearly independent eigenvectors)*

c) find eigenvectors and eigenvalues of this operator. *

1 Let $\{\mathbf{e}, \mathbf{f}\}$ be an orthonormal basis in $\mathbf{E}^2$. Consider the following ordered pairs:

a) $\{\mathbf{f}, \mathbf{e}\}$,

b) $\{\mathbf{f}, -\mathbf{e}\}$,

c) $\{\frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}, -\frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}\}$,

d) $\{\frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}, \frac{1}{2}\mathbf{e} - \frac{\sqrt{3}}{2}\mathbf{f}\}$.

Show that all these ordered pairs are orthonormal bases in $\mathbf{E}^2$.

Find amongst them the bases which have the same orientation as the orientation of the basis $\{\mathbf{e}, \mathbf{f}\}$.

Find amongst them the bases which have the orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}\}$.

2 Let $\{\mathbf{e}, \mathbf{f}\}$ be a basis in two-dimensional vector space V. Consider an ordered pair $\{\mathbf{a}, \mathbf{b}\}$ such that

$$\mathbf{a} = \mathbf{f},\ \ \mathbf{b} = \gamma\mathbf{e} + \mu\mathbf{f}\,,$$

where γ, μ are arbitrary real numbers.

Find values γ, μ such that an ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is a basis and this basis has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}\}$.

3 Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$ and let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.

4 Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider the following ordered triples:

a) $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$,

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* this question is just a recalling question.

b) $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$,
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$,
e) $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$,
f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$.

Show that all ordered triples a),b),c),d),e),f) are bases.
Show that the bases a), c), d) and f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, and the bases b) and e) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.
Show that bases d), e) and f) are orthonormal bases and bases a), b) and c) are not orthonormal bases.

5 Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in vector space V. Show that ordered triples $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$ and $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$ are bases and these bases have opposite orientations.

6 Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in 3-dimensional vector space V.
Consider in the space V the following ordered triples
I)—- $\{\mathbf{e} + 2\mathbf{f} + 3\mathbf{g}\,, 2\mathbf{f} + \mathbf{g}\,, \mathbf{e} + 2\mathbf{f} + \mathbf{g}\}$
II)—- $\{\mathbf{e} + \mathbf{f} - 2\mathbf{g}\,, 2\mathbf{f} + \mathbf{g}\,, \mathbf{e} + \mathbf{f} + \mathbf{g}\}$
III)—- $\{\mathbf{e} + 2\mathbf{f} + 4\mathbf{g}\,, \mathbf{e} + 3\mathbf{f} + 9\mathbf{g}\,, \mathbf{e} + 4\mathbf{f} + 16\mathbf{g}\}$
Show that all these oredered triples are bases.
Show that I-st and II-nd bases have opposite orientations.
Show that II-nd and III-d bases have the same orientations.
Show that I-st and III-nd bases have opposite orientations.