Homework 3

1 a) Show explicitly that matrix $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is an orthogonal matrix.

b) Show explicitly that under the transformation $\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\}A_\varphi$ an orthonormal basis transforms to an orthonormal one.

c) Show that for orthogonal matrix A_φ defined above the following relations are satisfied:

$$A_\varphi^{-1} = A_\varphi^T = A_{-\varphi}\,, \qquad A_\varphi \cdot A_\theta = A_{\varphi+\theta}\ .$$

2 Let $\mathbf{e}, \mathbf{f}$ be orthonormal basis in Euclidean space $\mathbf{E}^2$. Consider a vector

$$\mathbf{n}_\varphi = \mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi\,.$$

Let A be a linear orthogonal operator acting on the space $\mathbf{E}^2$ such that $A(\mathbf{e}) = \mathbf{n}_\varphi$.

We know that $\det A = \pm 1$ since A is orthogonal operator.

In the case if $\det A = 1$, find the image $A(\mathbf{f})$ of vector $\mathbf{f}$ and an image $A(\mathbf{x})$ of an arbitrary vector $\mathbf{x} = a\mathbf{e} + b\mathbf{f}$, write down the matrix of operator A in the basis $\mathbf{e}, \mathbf{f}$ and explain geometrical meaning of the operator A.

† How the answer will change if $\det A = -1$?

3 Let $\mathbf{e}, \mathbf{f}$ be an orthonormal basis in Euclidean space $\mathbf{E}^2$.

Consider a vector $\mathbf{N} = \mathbf{e} + \mathbf{f}$ in $\mathbf{E}^2$.

Let A be an orthogonal operator acting on the space $\mathbf{E}^2$ such that $A\mathbf{N} = \mathbf{N}$. ($\mathbf{N}$ is eigenvector of A with eigenvalue 1.) Suppose that A is not identity operator.

a) Find an action of operator A on the vector $\mathbf{R} = \mathbf{e} - \mathbf{f}$ in $\mathbf{E}^2$.

b) Write down the matrix of operator A in the basis $\mathbf{e}, \mathbf{f}$.

c) Explain geometrical meaning of the operator A.

4 Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in Euclidean space $\mathbf{E}^3$. Consider a linear operator P in $\mathbf{E}^3$ such that

$$\mathbf{e}' = P(\mathbf{e}) = \mathbf{e}, \quad \mathbf{f}' = P(\mathbf{f}) = \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}, \quad \mathbf{g}' = P(\mathbf{g}) = -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}\,.$$

Write down the matrix of operator P in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the order

Show that P is an orthogonal operator.

Show that orthogonal operator P preserves the orientation of $\mathbf{E}^3$.

Find an axis of the rotation and the angle of the rotation.

5 Consider a linear operator P_1 in $\mathbf{E}^3$ such that it transforms the orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ into the orthonormal basis $\{\mathbf{f}, \mathbf{e}, \mathbf{g}\}$:

$$P_1(\mathbf{e}) = \mathbf{f}\,, \quad P_1(\mathbf{f}) = \mathbf{e}\,, \quad P_1(\mathbf{g}) = \mathbf{g}\,.$$

Consider also a linear orthogonal operator P_2 such that it is the reflection operator with respect to the plane spanned by vectors $\mathbf{e}$ and $\mathbf{f}$.

Do operators P_1, P_2 preserve orientation?

Does operator $P = P_2 \circ P_1$ preserve orientation?

Find eignevector of operator P

Show that P is rotation operator.