Homework 0. Solutions

1 *Consider sets*

$$V = \{ax^2 + bx + c\,, a, b, c \in \mathbf{R}\}\,, \qquad T = \{x^2 + px + q\,, p, q \in \mathbf{R}\}$$

a) Explain why a set V is a vector space, and a set T is not a vector space (with respect to natural operations of mulitplication and addition of polynomials)

b) Explain why polynomials $1, x, x^2$ are linearly independent in V.

c) Calcualte dimension of V.

One can see that operations $+$ and $\cdot$ are well-defined: For two "vectors"—polynomials $P_1 = a_1x^2 + b_1x + c_1$ $P_2 = a_2x^2 + b_2x + c_2$

$$P_1 + P_2 = a_3x^2 + b_3x + c_3\,, \text{where } (a_3, b_3, c_3) = (a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1{+}_2, b_1 + b_2, c_1 + c_2)\,,$$

$$\lambda \cdot P_1 = \lambda()a_1x^2 + b_1x + c_1) = (a, b, c)\,, \text{ where } (a, b, c) = \lambda(a_1, b_1, c_1) = (\lambda a_1, \lambda b_1, \lambda c_1)\,.$$

We see that we may identify the space V with $\mathbf{R}^3$.

On the other hand T is not vector space, since if we consider two arbitrary polynomials in T their sum does not belong T,

Now prove that polynomials (vectors) $1, x, x^2$ are linearly independent. Let $c_1, c_2, c_3 \in \mathbf{R}$ be coefficinets such that

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0$$

i.e. polynomial $c_1{+}c_2x{+}c_3x^2$ is identically equal to zero. In this case it is equal at zero at points $x = 0, 1, -1$:

$$P(x) = c_1 + c_2x + c_3x^2 \equiv 0 \Rightarrow \begin{cases} P(0) = c_1 = 0 \\ P(1) = c_1 + c_2 + c_3 = 0 \\ P(-1) = c_1 - c_2 + c_3 = 0 \end{cases} \Rightarrow c_1 = 0\,, c_2 = 0\,, c_3 = 0\,,$$

i.e. polynomials $1, x, x^2$ are linearly independent.

2 *Show that the vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ in vector space V are linearly dependent if at least one of these vectors is equal to zero.*

WLOG suppose that $\mathbf{a}_1 = 0$. Then

$$\lambda\mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \ldots + 0 \cdot \mathbf{a}_n = 0$$

where λ is an arbitrary non-zero real number $\lambda \neq 0$. We see that there exists a linear combinations of vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ which is equal to zero and one of the coefficients $\{\lambda, 0, \ldots, 0\}$ is not equal to zero. Hence vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ are linearly dependent.

3 *a) Show that arbitrary three vectors in $\mathbf{R}^2$ are linearly dependent.*

Consider the following vectors in $\mathbf{R}^2$

$$\mathbf{e}_1 = (1, 0)\,, \qquad \mathbf{e}_2 = (0, 1)\,, \qquad \mathbf{a} = (2, 3)\,, \qquad \mathbf{b} = (3, 0)\,, \tag{1}$$

b) Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis in $\mathbf{R}^2$.

c) Show that $\{\mathbf{a}, \mathbf{b}\}$ is a basis in $\mathbf{R}^2$.

d) Show that $\{\mathbf{e}_1, \mathbf{b}\}$ is not a basis in $\mathbf{R}^2$.

`Solution of a)`

Consider arbitrary three vectors in $\mathbf{R}^2$

$$\mathbf{x}_1 = (a^1, a^2)$$
$$\mathbf{x}_2 = (b^1, b^2)$$
$$\mathbf{x}_3 = (c^1, c^2)$$

If vector $\mathbf{x}_1 = (a_1, a_2) = 0$ then nothing to prove. (See exercise 2). Let $\mathbf{x}_1 \neq 0$. WLOG suppose $a_1 \neq 0$. Consider vectors

$$\begin{array}{c} \mathbf{x}'_2 = \mathbf{x}_2 - \frac{b_1}{a_1}\mathbf{x}_1 = (b^1, b^2) - \frac{b_1}{a_1}(a_1, a_2) = (0, b'_2) \\ \mathbf{x}'_3 = \mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1 = (c^1, c^2) - \frac{c_1}{a_1}(a_1, a_2) = (0, c'_2) \end{array}$$

We see that vectors $\mathbf{x}'_2, \mathbf{x}'_3$ are proportional—i.e. they are linearly dependent: there exist $\mu_2 \neq 0$ or $\mu_3 \neq 0$ such that $\mu_2\mathbf{x}'_2 + \mu_3\mathbf{x}'_3 = 0$ E.g. we can take $\mu_2 = c'_2$, $\mu_3 = -b'_2$ in the case if $c'_2 \neq 0$ or $b'_2 \neq 0$ (if $c'_2 = b'_2 \neq 0$ then we can take coefficients μ_1, μ_2 any real numbers.) We have:

$$0 = \mu_2\mathbf{x}'_2 + \mu_3\mathbf{x}'_3 = \mu_2\left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1\right) + \mu_3\left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1\right) = \mu_2\mathbf{x}_2 + \mu_3\mathbf{x}_3 - \left(\frac{\mu_2 b_1}{a_1} + \frac{\mu_3 c_1}{a_1}\right)\mathbf{x}_1 = 0,$$

where $\mu_2 \neq 0$ or $\mu_3 \neq 0$. Hence vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent *. ■

`Solution of b)`
Vectors $\mathbf{e}_1, \mathbf{e}_2$ are linearly independent:

$$a\mathbf{e}_1 + b\mathbf{e}_2 = a(1, 0) + b(0, 1) = (a, b) = 0 \Rightarrow a = b = 0$$

We see that on one hand in $\mathbf{R}^3$ any trhee vectors are linearly dependent, and on the other hand there exist two linearly independent vectors. Hence dimension of $\mathbf{R}^2$ is equal to 2. Hence these two vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ form a basis

`Solution of c)` Vectors $\mathbf{a}, \mathbf{b}$ are also linearly independent:

$$x\mathbf{a} + y\mathbf{b} = x(2, 3) + y(3, 0) = (2x + 3y, 3x) = 0 \Rightarrow \begin{cases} x = 0 \\ 2x + 3y = 0 \end{cases} \Rightarrow x = y = 0\,.$$

We see that two vectors $\mathbf{a}, \mathbf{b}$ are linearly independent vectors in 2-dimensional space. Hence these two vectors $\{\mathbf{a}, \mathbf{b}\}$ form a basis

`Solution of d)` Vectors $\mathbf{e}_1, \mathbf{b}$ are linearly dependent, since

$$3\mathbf{e}_1 - \mathbf{b} = 0\,.$$

Hence this is not a basis.

4 *a) Show that* $\langle \mathbf{x}, \mathbf{y} \rangle = x^1y^1 + x^2y^2$ *does not define a scalar product in* $\mathbf{R}^3$.
b) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ *defines a scalar product in* $\mathbf{R}^3$.
c) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not define a scalar product in* $\mathbf{R}^3$.
f†*) Find necessary and sufficient conditions for entries* a, b, c *of symmetrical matrix* $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *such that the formula*

$$(\mathbf{x}, \mathbf{y}) = (\, x^1, x^2\,)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

defines a scalar product in $\mathbf{R}^2$.

`Solution`
Recall that scalar product on a vector space V is a function $B(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:
1) $B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$ (symmetricity condition)
2) $B(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \lambda B(\mathbf{x}, \mathbf{z}) + \mu B(\mathbf{y}, \mathbf{z})$ (linearity condition (with respect to the first argument))
3) $B(\mathbf{x}, \mathbf{x}) \geq 0$, $B(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0$ (positive-definiteness condition)
(The linearity condition with respect to the second argument follows from the conditions 2) and 1))

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* You may say: why so long proof? We know already that dimension of $\mathbf{R}^2$ is equal to 2 then by definition any three vectors in $\mathbf{R}^2$ have to be linear dependent. This "proof" is in fact "*circulus vicious* " since the proof of the fact that $\dim \mathbf{R}^2 = 2$ is founded on the statement of this exercise.

Remark Note that x^1, x^2, x^3—are components of the vector, do not be confused with exponents!

a) Show that $B(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$ *does not* define scalar product check the condition 3) of positive-definiteness: $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2$ may take zero values for $\mathbf{x} \neq 0$. E.g. if $\mathbf{x} = (0, 0, -1)$ $(\mathbf{x}, \mathbf{x}) = 0$, in spite of the fact that $\mathbf{x} \neq 0$. The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

b) Now show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is a scalar product in $\mathbf{R}^3$.

We need to check all the conditions above for scalar product for $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$:

1) $(\mathbf{y}, \mathbf{x}) = y^1x^1 + 3y^2x^2 + 5y^3x^3 = x^1y^1 + 3x^2y^2 + 5x^3y^3 = (\mathbf{x}, \mathbf{y})$. Hence it is symmetrical.

2) $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = (\lambda x^1 + \mu y^1)z^1 + 3(\lambda x^2 + \mu y^2)z^2 + 5(\lambda x^3 + \mu y^3)z^3 =$
$= \lambda(x^1z^1 + 3x^2z^2 + 5x^3z^3) + \mu(y^1z^1 + 3y^2z^2 + 5y^3z^3) = \lambda(\mathbf{x}, \mathbf{y}) + \mu(\mathbf{y}, \mathbf{z})$. Hence it is linear.

3) $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 \geq 0$. It is non-negative. If $\mathbf{x} = 0$ then obviously $(\mathbf{x}, \mathbf{x}) = 0$. If $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 = 0$, then $x^1 = x^2 = x^3 = 0$. Hence it is positive-definite.

All conditions are checked. Hence $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is indeed a scalar product in $\mathbf{R}^3$

c) Show that $B(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not* define scalar product check the condition 3): $(\mathbf{x}, \mathbf{x}) = 2x^1x^2 + (x^3)^2$ may take negative values. E.g. if $\mathbf{x} = (1, -1, 0)$ $(\mathbf{x}, \mathbf{x}) = -2 < 0$. The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

f) †)

The condition of linearity and symmetricity for the bilinear form

$$B(\mathbf{x}, \mathbf{y}) = (\, x^1, x^2 \,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

are evidently obeyed.

The general answer on this question is: symmetric matrix is positive-definite if and only if all principal minors are positive. For matrix under consideration it means that conditions $a > 0$ and $ac - b^2 > 0$ are necessary and sufficient conditions.

Give a proof for this special case.

Check the positive-definiteness condition.

For $\mathbf{x} = (1, 0)$ $B(\mathbf{x}, \mathbf{x}) = a$. Hence $a > 0$ is necessary condition. Now consider

$$B(\mathbf{x}, \mathbf{x}) = a(x^1)^2 + 2bx^1x^2 + c(x^2)^2 = \frac{(ax^1 + bx^2)^2 + (ac - b^2)(x^2)^2}{a} \geq 0 \Leftrightarrow ac - b^2 \geq 0$$

We see that $B(\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ iff $a > 0$ and $(ac - b^2) > 0$.
The positive-definiteness condition is id in fact the condition that discriminant of quadratic polynomial $at^2 + 2bt + c$ is non-positive, i.e. it does not take negative values (if $a > 0$).

5 *a) Let* $\mathbf{e}, \mathbf{f}$ *and* $\mathbf{g}$ *be three vectors in* 3*-dimensional Euclidean space* $\mathbf{E}^3$ *such that all these vectors have unit length and they are pairwise orthogonal.*

Show explicitly that the ordered set of these vectors $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is a basis.*

The space is 3-dimensional. Hence to show that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is a basis it suffices to show that vectors $(\mathbf{e}, \mathbf{f}, \mathbf{g})$ are linearly independent. Suppose $c_1\mathbf{e} + c_2\mathbf{f} + c_3\mathbf{g} = 0$. Take scalar product of this equation on the vector $\mathbf{e}$. Since vectors $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ have unit length and they are pairwise orthogonal then

$$(c_1\mathbf{e} + c_2\mathbf{f} + c_3\mathbf{g}, \mathbf{e}) = c_1(\mathbf{e}, \mathbf{e}) + c_2(\mathbf{f}, \mathbf{e}) + c_3(\mathbf{g}, \mathbf{e}) = c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 = c_1 = 0\,.$$

In the same way we prove that $c_2 = c_3 = 0$. Hence vectors $(\mathbf{e}, \mathbf{f}, \mathbf{g})$ are linearly independent.

6 *Let* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be three vectors in* 3*-dimensional Euclidean space* $\mathbf{E}^3$ *such that vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *have unit length, and are orthogonal to each other and vector* $\mathbf{c}$ *has length* $\sqrt{3}$ *and it forms an angle* $\varphi = \arccos\frac{1}{\sqrt{3}}$ *with vectors* $\mathbf{a}$ *and* $\mathbf{b}$.

Show that the ordered set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}-\mathbf{a}-\mathbf{b}\}$ *of vectors is an orthonormal basis in* $\mathbf{E}^3$*.*

Since vectors $\mathbf{a}$ and b have unit length and they are orthogonal to each other then $(\mathbf{a}, \mathbf{a}) = (\mathbf{b}, \mathbf{b}) = 1$ and $(\mathbf{a}, \mathbf{b}) = 0$. Since angle φ between vectors $\mathbf{a}$ and $\mathbf{c}$ equals to $\arccos \frac{1}{\sqrt{3}}$ and length of vector $\mathbf{c}$ equals to $\sqrt{3}$ then

$$(\mathbf{a}, \mathbf{c}) = |\mathbf{a}||\mathbf{c}| \cos \varphi = 1 \cdot \cdot \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 1.$$

Analogously $(\mathbf{b}, \mathbf{c}) = 1$ too. Hence scalar product of vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ with vector $\mathbf{a}$ equals to $(\mathbf{c}-\mathbf{a}-\mathbf{b}, \mathbf{a}) = 1-1-0 = 0$, i.e. vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ is orthogonal to the vector $\mathbf{a}$. In the same way we prove that vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ is orthogonal to the vector $\mathbf{b}$. Hence we proved that all vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}-\mathbf{a}-\mathbf{b}$ are pairwise orthogonal to each other. To see that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}-\mathbf{a}-\mathbf{b}\}$ is orthonormal basis it remains to prove that vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ is unit vector. This is the fact since

$$(\mathbf{c}-\mathbf{a}-\mathbf{b}, \mathbf{c}-\mathbf{a}-\mathbf{b}) = (\mathbf{c}, \mathbf{c}) + (\mathbf{a}, \mathbf{a}) + (\mathbf{b}, \mathbf{b}) - 2(\mathbf{c}, \mathbf{a}) - 2(\mathbf{c}, b) + 2(\mathbf{a}, \mathbf{b}) = \sqrt{3} \cdot \sqrt{3} + 1 + 1 - 2 \cdot 1 - 2 \cdot 1 = 1. \blacksquare$$

7 *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *be an orthonormal basis of Euclidean space* $\mathbf{E}^3$*. Consider the ordered set of vectors* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ *which is expressed via basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *in the following way:*

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$*;*

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$*;*

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$*;*

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ *(where* λ *is an arbitrary coefficient)?*

i) Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ *a basis in* $\mathbf{E}^3$*. Is this basis an orthonormal basis of* $\mathbf{E}^3$*?*

ii) Write down explicitly transition matrix which transforms the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *to the ordered set of the vectors* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$*. Is this matrix non-degenerate, or no? Is this matrix orthogonal?*

Answer question i) using the properties of corresponding transition matrices.

(you have to consider all cases a),b) c) and d)).

Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis in $\mathbf{E}^3$. Is this basis an orthonormal basis of $\mathbf{E}^3$?

(you have to consider all cases a),b) c) and d)).

To analyse the cases we use the definition of basis: 3 vectors in 3-dimensional space form a basis if and only if these vectors are linearly independent.

Case a) Vectors $\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \mathbf{e}'_3 = \mathbf{e}_3$ are linearly independent, since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis. Hence $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis too. Al vectors of this basis have unit lenght and they are orthogonal to each other. Hence this is orthonormal basis.

Case b) Vectors $\mathbf{e}'_1 = \mathbf{e}_1, \mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}'_3 = \mathbf{e}_3$ are linearly dependent. Indeed

$$\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = \mathbf{e}_1 - (\mathbf{e}_1 + 3\mathbf{e}_3) + 3\mathbf{e}_3 = 0\,.$$

Hence it is not a basis.

Case c) First two vectors $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$ are already linearly dependent: $\mathbf{e}'_1 = 3\mathbf{e}'_2$. Hence these three vectors do not form a basis.

Case d) Check are vectors linearly independent or not. Let $c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = 0$, i.e.

$$c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = c_1\mathbf{e}_2 + c_2\mathbf{e}_1 + c_3(\mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3) = (c_2 + c_3)\mathbf{e}_1 + (c_1 + c_3)\mathbf{e}_2 + c_3\lambda\mathbf{e}_3 = 0\,.$$

I-st case $\lambda \neq 0$. We have $c_2 + c_3 = c_1 + c_3 = \lambda c_3 = 0$. Hence $c_3 = 0, c_1 = 0, c_2 = 0$. These three vectors are linearly independent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis.

II-nd case $\lambda = 0$. We have $c_2 + c_3 = c_1 + c_3 = 0c_3 = 0$. Hence c_3 can be an arbitrary number and $c_1 = -c_3, c_2 = -c_3$. c_3 These three vectors are linearly dependent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis.

Now we will answer the same questions using transition matrices.

Case a) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$ is evidently orthonormal basis since all vectors have unit length, and they are orthogonal to each other. We answered this question above. Now we will answer this question again, using transition matrix.

Calculate the transition matrix: Transition matrix $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_3\} T\,.$$

This matrix is non-degenerate, its rank is equal to 3 ($\det T = 1 \neq 0$). It is orthogonal matrix, because all the rows have unit length, and are orthogonal to each other.

Since transition matrix from orthonormal basis $\{\mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_3\}$, matrix T is an orthogonal matrix, hence the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, ($\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} T$) is on orthonormal basis also. We come to the same answer as in the previous exercise.

Case b) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = 0$. We answered this question above. Now we answer ths question using transition matrix.

Calculate the transition matrix. Transition matrix $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}$,

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} T\,.$$

This matrix is degenerate, its rank ≤ 2. One can see it noting that rows are linear dependent or noting that $\det T = 0$. Vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are linear dependent. On the other hand vectors $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ are linear independent. Hence rank of the matrix T is equal to 2, and this matrix is non-degenerate (its determinant vanishes). Of course this matrix is not orthogonal— orthogonal matrix has to be at least non-degenerate. (determinant of orthogonal matrix is equal to ± 1).

Since transition matrix from basis $\{\mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_3\}$, matrix T is degenerate matrix ($\det T = 0$), hence the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, ($\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} T$) is a row of linear dependent vectors, this is not a basis. We come to the same answer as in the previous exercise.

Case c) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1 - \mathbf{e}_2, 3\mathbf{e}_1 - 3\mathbf{e}_2, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $3\mathbf{e}'_1 - \mathbf{e}'_2 = 0$. We answered this question above. One can see it also studying the transition matrix.

Transition matrix $T = \begin{pmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} T\,.$$

This matrix is degenerate: the second column is proportional to the first column, ($\det T = 0$.) Hence the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, ($\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} T$) is a row of linear dependent vectors, this is not a basis. We come to the same answer as in the previous exercise.

(Rank of the matrix rank ≤ 2. On the other hand second and third column of this matrix are linear independent. Hence rank of the matrix T is equal to 2).

Case d)

The transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \lambda \mathbf{e}_3\}$ is $T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) T$

I-st case. $\lambda \neq 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis because vectors are linear independent. This basis is not orthogonal, because the length of vector $\mathbf{e}'_3$ is not equal to 1 ($(\mathbf{e}'_3, \mathbf{e}'_3) = |\mathbf{e}'_3|^2 = 2 + \lambda^2$). This matrix is not orthogonal, because the new basis is not orthonormal.

II-nd case $\lambda = 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis because vectors are linear independent: $\mathbf{e}'_1 + \mathbf{e}'_2 - \mathbf{e}'_3 = 0$.

We answered this question in the previous exercise. Now we will answer yousing the transition matrix. Calculate deterinant of the transition matrix: $\det T = -\lambda$. If $\lambda \neq 0$, the matrix T is non-degenerate, and the ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis, but this basis is not orthonormal, because the matrix T is not orthogonal (the second row of this matrix has not unit length.)

If $\lambda = 0$, the matrix T is degenerate, the vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ are linearly dependent, thus the ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis. (The transition matrix T has rank less or equal to 2, because vectors are linear dependent. On the other hand vectors $\mathbf{e}'_1, \mathbf{e}'_2$ are linear independent. Hence the rank of the matrix is equal to 2.)

8† *Prove the Cauchy–Bunyakovsky–Schwarz inequality*

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}),$$

where $\mathbf{x}, \mathbf{y}$ *are arbitrary two vectors and* (,) *is a scalar product in Euclidean space.*

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ *consider the quadratic polynomial* $At^2 + 2Bt + C$ *where* $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})$. *Show that this polynomial has at most one real root and consider its discriminant.*

Consider quadratic polynomial $P(t) = \sum_{i=1}^n (tx^i + y^i)^2 = At^2 + 2Bt + C$, where $A = \sum_{i=1}^n (x^i)^2 = (\mathbf{x}, \mathbf{x})$, $B = \sum_{i=1}^n (x^i y^i) = (\mathbf{x}, \mathbf{y})$, $C = \sum_{i=1}^n (y^i)^2 = (\mathbf{y}, \mathbf{y})$. We see that equation $P(t) = 0$ has at most one root (and this is the case if only vector $\mathbf{x}$ is collinear to the vector $\mathbf{y}$). This means that discriminant of this equation is less or equal to zero. But discriminant of this equation is equal to $4B^2 - 4AC$. Hence $B^2 \leq AC$. It is just CBS inequality. ($(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})$, i.e. discriminant is equal to zero $\Leftrightarrow$ vectors $\mathbf{x}$, $\mathbf{y}$ are colinear.