Homework 2. Solutions

0[*] *Let P be an operator in 2-dimensional vector space V such that in the given basis $\mathbf{e}_1, \mathbf{e}_2$ the matrix of this operator is*

$$\begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}.$$

a) write down the action of the operator P on vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

b) Show without long calculations that matrix of this operator in the 'suitable' basis is $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. (You may use without a proof that this operator indeed has two eigenvectors)

c) find eigenvectors and eigenvalues of this operator.

a) $P(\mathbf{e}_1) = 5\mathbf{e}_1 + 2\mathbf{e}_2$, $P(\mathbf{e}_2) = -\mathbf{e}_1 + 2\mathbf{e}_2$.

b) Let $\mathbf{a}, \mathbf{b}$ be eigenvectors of this operator: $P(\mathbf{a}) = \lambda\mathbf{a}, P(\mathbf{b}) = \mu\mathbf{b}$. Thus a matrix of the operator P in the basis $\mathbf{a}, \mathbf{b}$ is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. On the other hand $\det P = 5 \cdot 1 - 2 \cdot (-1) = 12 = \lambda\mu$ and $\operatorname{Tr} P = 5 = 2 = \lambda + \mu$, hence $\begin{cases} \lambda = 4 \\ \mu = 3 \end{cases}$ (or $\begin{cases} \lambda = 3 \\ \mu = 4 \end{cases}$).

c) we also know eigenvalues. Find eigenvectors $\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2$ and $\mathbf{b} = b^1\mathbf{e}_1 + b^2\mathbf{e}_2$.

1) $\lambda = 3$:

$$P\mathbf{a} = 4\mathbf{a}, \ \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = 4 \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \Rightarrow \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix},$$

and

$$P\mathbf{b} = 3\mathbf{b}, \ \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = 4 \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} \Rightarrow \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{pmatrix} s \\ 2s \end{pmatrix},$$

1 *Let $\{\mathbf{e}, \mathbf{f}\}$ be an orthonormal basis in $\mathbf{E}^2$. Consider the following ordered pairs:*

a) $\{\mathbf{f}, \mathbf{e}\}$

b) $\{\mathbf{f}, -\mathbf{e}\}$

c) $\{\frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}, -\frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}\}$

d) $\{\frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}, \frac{1}{2}\mathbf{e} - \frac{\sqrt{3}}{2}\mathbf{f}\}$

Show that all these ordered pairs are orthonormal bases in $\mathbf{E}^2$.

Find amongst them the bases which have the same orientation as the orientation of the basis $\{\mathbf{e}, \mathbf{f}\}$.

Find amongst them the bases which have the orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}\}$.

Solution:

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[*] *this question is just a recalling question.*

First check that all the bases are orthonormal. For the bases a) and b) this is obvious: both vectors have unit length and they are orthogonal to each other.

Check orthogonality condition for the basis d). (For the basis a) all calculations are analogous). We have to check that vectors $\mathbf{a} = \frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}$ and $\mathbf{b} = \frac{1}{2}\mathbf{e} - \frac{\sqrt{3}}{2}\mathbf{f}$ have both unit length and are orthogonal to each other. For calculations we use the fact that initial basis is orthonormal too, i.e. vectors $\mathbf{e}, \mathbf{f}$ have unit length: scalar products $(\mathbf{e}, \mathbf{e})$, $(\mathbf{f}, \mathbf{f})$ both are equal to 1 and these vectors are orthogonal: scalar product $(\mathbf{e}, \mathbf{f})$ is equal to zero. Calculate scalar products:

$$(\mathbf{a}, \mathbf{a}) = \left(\frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}, \frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e}, \mathbf{e}) + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{f}, \mathbf{e}) + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}(\mathbf{e}, \mathbf{f}) + \frac{1}{2}\cdot\frac{1}{2}(\mathbf{f}, \mathbf{f}) =$$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0 + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{3}{4} + \frac{1}{4} = 1. \quad (1)$$

We see that scalar product $(\mathbf{a}, \mathbf{a})$ is equal to 1. This means that $|\mathbf{a}| = 1$.

Analogously we show that the length of the vector $\mathbf{b}$ is equal to 1:

$$(\mathbf{b}, \mathbf{b}) = \left(\frac{1}{2}\mathbf{e} - \frac{\sqrt{3}}{2}\mathbf{f}, \frac{1}{2}\mathbf{e} - \frac{\sqrt{3}}{2}\mathbf{f}\right) = \frac{1}{2}\cdot\frac{1}{2}\cdot 1 - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 0 - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0 + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 = 1\,. \quad (2)$$

It remains to show that these vectors are orthogonal , i.e. their scalar product is equal to zero:

$$(\mathbf{a}, \mathbf{b}) = \left(\frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}, \frac{1}{2}\mathbf{e} - \frac{\sqrt{3}}{2}\mathbf{f}\right) = \frac{\sqrt{3}}{2}\cdot\frac{1}{2}(\mathbf{e}, \mathbf{e}) + \frac{1}{2}\cdot\frac{1}{2}(\mathbf{f}, \mathbf{e}) + -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e}, \mathbf{f}) - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{f}, \mathbf{f}) =$$

$$\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2}\cdot 0 + -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0 - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 = 0, \quad (3)$$

i.e. these vectors are orthogonal

Orthonormality conditions for the basis a) could be checked in the same way.

Remark You could ask a question: how comes we call these pairs bases without checking the condition that they are the bases. The point is that if two vectors are not equal to zero and are orthogonal each other (and this was checked) this implies that they are not linearly dependent. (Why?: see the exercise 5) in Homework 0. Hence the ordered pair of these two vectors form a basis.

Now find orientation of these bases with respect to the basis $\{\mathbf{e}, \mathbf{f}\}$. (We already show that all ordered pairs are bases.)

Case a) One can easy see that transition matrix from the basis $\{\mathbf{e}, \mathbf{f}\}$ to the basis $\{\mathbf{f}, \mathbf{e}\}$ is

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1a))$$

Indeed $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f})T$: $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Calculate determinant of transition matrix $\det T = -1 < 0$. Hence this basis has an orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis. Of course in this case it is obvious.).

Case b) Analogously for the case b) one can easy see that transition matrix from the basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\mathbf{f},-\mathbf{e}\}$ is

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Indeed $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f})T$: $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Calculate determinant of transition matrix $\det T = 1 > 0$. Hence this basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f}\}$. (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis. Of course in this case it is obvious.)

Case c) Analogously for the case c) one can easy see that transition matrix from the basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\frac{\sqrt{2}}{2}\mathbf{e}+\frac{\sqrt{2}}{2}\mathbf{f}, -\frac{\sqrt{2}}{2}\mathbf{e}+\frac{\sqrt{2}}{2}\mathbf{f}\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Calculate determinant of transition matrix $\det T = 1 > 0$. Hence this basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f}\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis.)

Case d) and finally for the case d) one can easy see that transition matrix from the basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\frac{\sqrt{3}}{2}\mathbf{e}+\frac{1}{2}\mathbf{f}, \frac{1}{2}\mathbf{e}-\frac{\sqrt{3}}{2}\mathbf{f}\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

Calculate determinant of transition matrix $\det T = -1 < 0$. Hence this basis has an orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis.)

2*Let* $\{\mathbf{e},\mathbf{f}\}$ *be a basis in two-dimensional linear space* V*. Consider an ordered pair* $\{\mathbf{a},\mathbf{b}\}$ *such that*

$$\mathbf{a} = \mathbf{f},\ \mathbf{b} = \gamma\mathbf{e} + \mu\mathbf{f}\,,$$

where γ, μ are arbitrary real numbers.

Find values γ, μ such that an ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is a basis and this basis has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}\}$.

Solution: Transition matrix T from the basis $\{\mathbf{e}, \mathbf{f}\}$ to the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is matrix $T = \begin{pmatrix} 0 & \gamma \\ 1 & \mu \end{pmatrix}$: $\{\mathbf{a}, \mathbf{b}\} = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} 0 & \gamma \\ 1 & \mu \end{pmatrix}$. Its determinant is equal to $-\gamma$. Hence the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is a basis if and only if $\gamma \neq 0$. (This can be done in other way: one can see that if $\gamma = 0$ then vectors $\mathbf{a} = \mathbf{f}$ and $\mathbf{b} = \mu\mathbf{f}$ are linear dependent, and If $\gamma \neq 0$ then vectors $\mathbf{a} = \mathbf{f}$ and $\mathbf{b} = \mu\mathbf{f}$ are linear independent.)

If $\det T = -\gamma > 0$, i.e. $\gamma < 0$ then the bases $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{e}_x, \mathbf{e}_y\}$ have the same orientation.

If $\det T = -\gamma < 0$, i.e. $\gamma > 0$ then the bases $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{e}_x, \mathbf{e}_y\}$ have opposite orientation.

3 *Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.*

Solution:. Bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientation since $(\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T$, where $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\det T = -1 < 0$.

Hence an arbitrary basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the equivalence class of of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ (with respect to orientation relation) or to the equivalence class of of the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.(See the Proposition in the subsection 1.9 of Lecture notes).

We repeat the proof of the proposition for this special case:

Let T_1 be transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and T_2 be transition matrix from the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1, \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2\,.$$

We see that $T_1 = T \cdot T_2$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)TT_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1 \textit{ i.e. } TT_2 = T_1\,.$$

If $\det T_2 > 0$ then the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$. If $\det T_2 < 0$ then $\det T_1 = \det(TT_2) = \det T \cdot \det T_2 > 0$ because $\det T = -1 < 0$. Hence in this case the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

In other words bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientations. Hence they belong to different classes of bases (with respect to orientation). There are two

classes. Hence the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ or the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.

Arbitrary basis has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ or vice versa: it has the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

4 *Let* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ *be an orthonormal basis in* $\mathbf{E}^3$. *Consider the following ordered triples:*

a) $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$,
b) $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$,
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$,
e) $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$,
f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$.

Show that all these ordered triples a),b),c),d),e),f) are bases.
Show that the bases a), c), d) and f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, *and the bases b) and e) have the orientation opposite to the orientation of the basis* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

Show that bases d), e) and f) are orthonormal bases and bases a), b) and c) are not orthonormal bases.

Solution:

Recall that to check that is an ordered triple a basis or no, and to find an orientation of this basis we have to find transition matrix T. If this matrix is non-degenerate, i.e. $\det T \neq 0$ then it transforms basis to a basis. If determinant of transition matrix is positive, then these two bases have the same orientation. If determinant of transition matrix is negative , then these two bases have opposite orientation.

To show that basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis we have to show that all basis vectors have the unit length and they are orthogonal each other, i.e. we have to check that following relations are satisfied:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{4.1}$$

(Another way to check orthonormality of new basis: one have to check that transition matrix is orthogonal, i.e. it satisfies the condition $TT^t = I$ (I is identity matrix), i.e. it is a orthogonal matrix.)

Case a). One can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triples $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad (1a))$$

Indeed $\{\mathbf{e}_x, \mathbf{e}_x+2\mathbf{e}_y, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}T$: $\{\mathbf{e}_x, \mathbf{e}_x+2\mathbf{e}_y, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Calculate determinant of transition matrix: $\det T = 10 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$ is a basis. $\det T = 10 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case b). In this case analogously: transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = -5 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$ is a basis. $\det T = -5 < 0$. Hence this new basis has the orientation opposite to the orientation of the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case c) Analogously: transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 5 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$ is a basis. $\det T = 5 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case d) Transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } \left\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\right\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is a basis. $\det T = 1 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis orthonormal basis. Indeed the first two vectors have the length 1 and they are orthogonal to each other (See equations (1,2,3) in the previous exercise). The third vector $\mathbf{e}_z$ has length one and it is obviously is orthogonal to first two vectors.

Case e) for the case e) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is a basis. $\det T = 1 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is obviously orthonormal basis because all the vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ have unit length and they are orthogonal to each other.

Case f) for the case f) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ is a basis. $\det T = -1 > 0$. Hence this new basis has the orientation opposite to the orientation of the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. (Another way to see it: we come to basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ from the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ just by swapping the vectors $\mathbf{e}_x$ and $\mathbf{e}_y$.) This new basis is obviously orthonormal basis because all the vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ have unit length and they are orthogonal to each other.

5 *Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in linear three-dimensional space V.*
Consider the following ordered triples:

$$\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}, \quad \{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$$

Show that these ordered triples are bases and these bases have opposite orientations.

To write transition matrix from the basis $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$ to the basis $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$? This is little bit long exercise. We do it in another way:

Consider transition matrices;

T_1—transition matrix from the initial basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ to the basis $\{\mathbf{f},\mathbf{e}+2\mathbf{f},3\mathbf{g}\}$.

T_2—transition matrix from the initial basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ to the basis $\{\mathbf{e},\mathbf{f},2\mathbf{f}+3\mathbf{g}\}$ It is easy to see that

$$T_1=\begin{pmatrix}0&1&0\\1&2&0\\0&0&3\end{pmatrix}\qquad\text{and}\qquad T_2=\begin{pmatrix}1&0&0\\0&1&2\\0&0&3\end{pmatrix}$$

We see that determinant of the transition matrix T_1 is negative and the determinant of the transition matrix T_2 is positive. This means that the both ordered triples $\{\mathbf{f},\mathbf{e}+2\mathbf{f},3\mathbf{g}\}$ and $\{\mathbf{e},\mathbf{f},2\mathbf{f}+3\mathbf{g}\}$ are bases. The first basis has an orientation opposite to the orientation of the initial basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ and the second basis has the same orientation as the initial basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$. Hence these both bases have opposite orientation

6 *Let* $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ *be a basis in* 3*-dimensional vector space* V*.*
Consider in the space V *the following ordered triples*
I)—- $\{\mathbf{e}+2f+3\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+2\mathbf{f}+\mathbf{g}\}$
II)—- $\{\mathbf{e}+\mathbf{f}-2\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+\mathbf{f}+\mathbf{g}\}$
III)—- $\{\mathbf{e}+2\mathbf{f}+4\mathbf{g}\,,\mathbf{e}+3\mathbf{f}+9\mathbf{g}\,,\mathbf{e}+4f\mathbf{f}+16\mathbf{g}\}$
Show that all these oredered triples are bases.
Show that I-st and II-nd bases have opposite orientations.
Show that II-nd and III-d bases have same orientations.
Show that I-st and III-nd bases have opposite orientations.

Calculate transition matrices T_I, T_{II} and T_{III} from intial basis $\mathbf{e},\mathbf{f},\mathbf{g}$ to the these triples:

$$\{\mathbf{e}+2f+3\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+2\mathbf{f}+\mathbf{g}\}=\{\mathbf{e},\mathbf{f},\mathbf{g}\}\underbrace{\begin{pmatrix}1&0&1\\2&2&2\\3&1&1\end{pmatrix}}_{T_I},\qquad \det T_I=-4\,,$$

$$\{\mathbf{e}+f-2\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+\mathbf{f}+\mathbf{g}\}=\{\mathbf{e},\mathbf{f},\mathbf{g}\}\underbrace{\begin{pmatrix}1&0&1\\1&2&1\\-2&1&1\end{pmatrix}}_{T_I},\qquad \det T_I=6\,,$$

$$\{\mathbf{e}+2f+4\mathbf{g}\,,\mathbf{e}+3\mathbf{f}+9\mathbf{g}\,,\mathbf{e}+4\mathbf{f}+16\mathbf{g}\}=\{\mathbf{e},\mathbf{f},\mathbf{g}\}\underbrace{\begin{pmatrix}1&1&1\\2&3&4\\4&9&16\end{pmatrix}}_{T_I},\qquad \det T_I=2\,.$$

We see that all transition matrices are not degenerate, hence all the triples are bases.

The first transition matrix has negative determinant, hence the I-st basis has orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}, \}$. Respectively the second transition matrix has positive determinant, hence the II-nd basis has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}, \}$. We see that I-st and II-nd bases belong to different equivalence classes. Hence they have opposite orientations. The third transition matrix has positive determinant. Hence II-nd and III-rd bases both have the same orientation as initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. Thus we see that II-nd and III-rd bases have the same orientation.

Finally since first basis has orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and III-rd basis has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ hence I-st and III-rd bases have opposite orientations.