Homework 4. Solutions

1 *a) Let P be an orthogonal operator in $\mathbf{E}^3$, and let $\mathbf{a} \neq 0$ be its eigenvector: $P\mathbf{a} = \lambda\mathbf{a}$. Show that the eigenvalue $|\lambda| = 1$. b) suppose that an orthogonal operator P which also preserves orinetation of $\mathbf{E}^3$, and for orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$,*

$$P(\mathbf{e}) = \mathbf{f}\,, P(\mathbf{g}) = -\mathbf{g}\,.$$

Find the action of this operator on arbitrary vector in $\mathbf{E}^3$.

Explain why this operator is rotation operator. Find axis and angle of the rotation.

Solution a) Orthogonal operator preserves scalar product: $(\mathbf{a}, \mathbf{a}) = (P\mathbf{a}, P\mathbf{a}) = \lambda^2(\mathbf{a}, \mathbf{a})$. This implies that $|\lambda| = 1$.

b) Matrix of the operator P in orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is $\begin{pmatrix} 0 & x & 0 \\ 1 & y & 0 \\ 0 & z & -1 \end{pmatrix}$ The matrix is orthogonal one. Hence columns are orthogonal to each other: taking scalar (dot) product of the first and second column we will come to $y = 0$, and taking scalar (dot) product of the third and second column we will come to $z = 0$, Length of the second column is equal to 1, hence $x = \pm 1$. $\det P = x$ hence $x = 1$ since P preserves orientation.

We see that

$$P(\mathbf{e}) = \mathbf{f}\,, \quad , P(\mathbf{f}) = \mathbf{e}\,, \quad P(\mathbf{g}) = -\mathbf{g}\,.$$

Hence $P(\mathbf{e}+\mathbf{f}) = \mathbf{e}+\mathbf{f}$, i.e. $\mathbf{N} = \mathbf{e}+\mathbf{f}$ is eigenvector woth eigenvalue $\lambda = 1$. The axis of rotation goes along the vector $\mathbf{N}$. The angle of rotation is π since $\operatorname{Tr} P = 1 + 2\cos\varphi = -1$.

2 *Consider an operator P such that*

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

where $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that it is orthogonal operator preserving orientation.

Show that this operator defines rotation, and find the axis and the angle of this rotation.

It is easy to see that

$$\begin{array}{rcr}
(\mathbf{e}', \mathbf{e}') = (P(\mathbf{e}), P(\mathbf{e})) = & \left(\frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}, \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\right) = & 1, \\
(\mathbf{e}', \mathbf{f}') = (P(\mathbf{e}), P(\mathbf{f})) = & \left(\frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}, -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\right) = & 0, \\
(\mathbf{e}', \mathbf{g}') = (P(\mathbf{e}), P(\mathbf{g})) = & \left(\frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\right) = & 0, \\
(\mathbf{f}', \mathbf{f}') = (P(\mathbf{f}), P(\mathbf{f})) = & \left(-\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}, -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\right) = & 1, \\
(\mathbf{f}', \mathbf{g}') = (P(\mathbf{f}), P(\mathbf{g})) = & \left(-\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\right) = & 0, \\
(\mathbf{g}', \mathbf{g}') = (P(\mathbf{g}), P(\mathbf{g})) = & \left(-\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\right) = & 1
\end{array}$$

new basis is orthonormal one. Hence P is orthogonal operator. The matrix of operator P is

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}$$

(One can check straightforwarly that matrix of the operator P in orthonormal baasis is orthogonal matrix,) The determinant of P equals to $\det P = 1$. Operator P preserves orientation. To find an axis we have to find eigenvector of this matrix with eigenvalue 1. Eigenvalue equals 1, since this is rotation: We have

$$P\mathbf{N} = \mathbf{N}, \quad \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\,.$$

Solving these equations we come to $x = y = -z$, i.e. $\mathbf{N}$ is proportional to the vector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. , i.e. axis is directed along the vector $\mathbf{N} = \mathbf{e} + \mathbf{f} - \mathbf{g}$.

Trace of the operator P is equal to $\mathrm{Tr}\, P = \frac{1}{3}(2+2+2) = 2 = 1+2\cos\varphi$. Hence $\cos\varphi = \frac{1}{2}$, i.e. $\varphi = \pm\frac{\pi}{3}$.

We see that operator P defines rotation with respect to axis directed along the vector $\mathbf{e}+\mathbf{f}-\mathbf{g}$ on the angle $\pm\frac{\pi}{3}$.

3 *Consider on* $\mathbf{E}^3$ *following two operators:*

$$P_1(\mathbf{x}) = \mathbf{x} - 2(\mathbf{n},\mathbf{x})\mathbf{n}\,, \quad P_2(\mathbf{x}) = 2(\mathbf{n},\mathbf{x})\mathbf{n} - \mathbf{x}\,,$$

where $\mathbf{n}$ *is a unit vector.*

Show that these both operators are orthogonal operators. Show that first operator changes the orientation, and the second operator preserves orientation.

Show that the first operator is reflection operator with respect to....

Show that the second operator is rotation operator: find an axis of rotation and an angle of rotation.

One can see that for operator P_1

$$P_1(\mathbf{n}) = \mathbf{n} - 2(\mathbf{n},\mathbf{n})\mathbf{n} = -\mathbf{n}$$

since $(\mathbf{n},\mathbf{n}) = 1$, and for an arbitrary vector $\mathbf{y}$ belonging to the plane $\alpha_{\mathbf{n}}$ which is orthogonal to the vector $\mathbf{n}$

$$P_1(\mathbf{y}) = \mathbf{y} - 2(\mathbf{n},\mathbf{y})\mathbf{n} = \mathbf{y}$$

since $(\mathbf{n},\mathbf{y}) = 0$. We see that operator P is identical on the plane orthogonal to $\mathbf{n}$ and $\mathbf{n} \mapsto -\mathbf{n}$. Hence P_1 is orthogonal operator and it is reflection operator with respect to the plane $\alpha_{\mathbf{n}}$.

Operator $P_2 = -P_1$, Operator P_1 changes orientation, hence operator P_2 preserves orientation: $\det P_2 = -\det P_1 > 0$. Hence it is orthogonal operator, preserving orientation i.e. rotation operator. One can see that

$$P_2(\mathbf{n}) = -P_1(\mathbf{n}) = \mathbf{n}$$

Axis of this operator is directed along vector $\mathbf{n}$. Arbitrary vector $\mathbf{y}$ belonging to the plane α_y is multiplied on -1 (is eigenvector with eigenvalue -1):

$$P_2(\mathbf{y}) = -P_1(\mathbf{y}) = -\mathbf{y}$$

This means that plane $\alpha_{\mathbf{n}}$ rotates on the angel π.

Remark One can do it using brute force: calculate the matrix of operator and convince that its determinant equals 1. But calculations in general case are long and boring: E.g. matrix of operator P_2 has the foolowing appearance:

$$\begin{pmatrix} 2n_x^2 - 1 & 2n_x n_y & 2n_x n_z \\ 2n_y n_x & 2n_y^2 - 1 & 2n_y n_z \\ 2n_z n_x & 2n_z n_y & 2n_z^2 - 1 \end{pmatrix}$$

, where $\mathbf{n} = (n_x, n_y, n_z)$!?.).

Vector $\mathbf{n} = (n_x, n_y, n_z)$ is eigenvector with eigenvalue 1 and Trace of this matrix is equal to 1. Thus we will come to the same answer, but claculations are really much more complicated... (Just try to check that this matrix is orthogonal!)

4 *Orthogonal operator* P *obeys the condition*

$$P \neq \mathbf{id},, \quad \text{and} \quad P^5 = \mathbf{id}\,.$$

Show that P *is rotation operator, and calculate the angle of rotation.*

$P^5 = 1$. take determinant: $\det P^5 = (\det P)^5 = 1$, i.e. $\det P = 1$. We see that orthogonal operator P preserves orientation. If angle of rotation is equal to φ, then $5\varphi = 2\pi$, i.e. $\varphi = \frac{2\pi k}{4}$, whjere $k = 1,2,3,4$, $(P \neq 1)$.

5 *Students John and Sarah calculate vector product* $\mathbf{a}\times\mathbf{b}$ *of two vectors using two different orthonormal bases in the Euclidean space* $\mathbf{E}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. *John expands the vectors with respect to* *the orthonormal basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$*. Sarah expands the vectors with respect to the basis* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$*. For two arbitrary vectors* $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a'_1\mathbf{e}'_1 + a'_2\mathbf{e}'_2 + a'_3\mathbf{e}'_3\,,$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b'_1\mathbf{e}'_1 + b'_2\mathbf{e}'_2 + b'_3\mathbf{e}'_3\,.$$

John and Sarah both use so called "determinant" formula. Are their answers the same?

$$\mathbf{a} \times \mathbf{b} = \underbrace{\det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}_{\textit{John's calculations}} \stackrel{?}{=} \underbrace{\det\begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{pmatrix}}_{\textit{Sarah's calculations}}$$

Solution: In the case if bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ have the same orientation, then answer will be the same. If bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, e'_3\}$ have opposite orientation then the answer of John will differ from the answer of Sarah by sign. Explain why.

Let third student, say David enters, the "game". David knows that formulae of John and Sarah both obey to axioms defining vector product (see the lecture notes). Without paying attention on formulae of John and Sarah he just uses the axioms defining vector product: He will consider the direction orthogonal to the plane spanned by vectors $\mathbf{a}, \mathbf{b}$ and take the vector such that its length equals the area of parallelogram. One thing that David also have to do it is to choose the direction of this vector. It is here where the question of orientation of bases becomes crucial.

Suppose David uses an orthonoromal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ defining the orientation, which has the same orientation as the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which John uses.

According of the fifth axiom he chooses the direction of the vector $\mathbf{c}$ in a such way that bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ have the same orientation.

Now the answer is clear: if bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (of John) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) have the same orientation then all three bases of David, John and Sarah will have the same orientation, hence all three answers will coincide: all bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ (calculation of vector product), $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ (David's basis) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (John's basis) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (Sarah's basis) have the same orientation.

If bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (of John) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) have opposite orientation then answer of David will coincide with answer of John and it will have the opposite sign with answer of Sarah:

Indeed in this case the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ (David's basis) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (John's basis) will have the same orientation, hence the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) will have opposite orientation. Hence calculations of vector product in the basis which Sarah is using lead to the answer $-\mathbf{c}$: in this case the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) will have the same orientation. *

6 *Let* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *be an orthonormal basis in* $\mathbf{E}^3$ *Find a vector* $\mathbf{n}$ *in* $\mathbf{E}^3$*, such that the following conditions hold:*

1) It has unit length

2) It is orthogonal to the vectors $\mathbf{a} = \mathbf{e} + 2\mathbf{f} + 3\mathbf{g}$ *and* $\mathbf{b} = \mathbf{e} + 3\mathbf{f} + 2\mathbf{g}$.

3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ *has an orientation opposite to the orientation of the basis* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$

(You have to expand vector $\mathbf{n}$ *over the basis* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$*).*

Solution: Consider a vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$. We define orientation in $\mathbf{E}^3$ by the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. The vector $\mathbf{N}$ is orthogonal to vectors $\mathbf{a}, \mathbf{b}$ (vector product) and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$ is a unit vector. It remains to solve the problem of orientation. Both vectors $\pm\frac{\mathbf{N}}{|\mathbf{N}|}$ are unit vectors which are orthogonal to vectors $\mathbf{a}, \mathbf{b}$. On the other hand the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis and this basis has the same orientation as a basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. This follows from the axioms defining the vector product and the fact that vectors $\mathbf{N} \neq 0$, i.e. the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis. Hence the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ where $\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|}$ has an orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

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* In the case if one of vectors equals zero and vectors do not span plane then on can see that all three students John, Sarah and David will come to the answer: zero.

The vector

$$\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|} = -\frac{(\mathbf{e}_x + 2\mathbf{e}_y + 3\mathbf{e}_z) \times (\mathbf{e}_x + 3\mathbf{e}_y + 2\mathbf{e}_z)}{|\mathbf{N}|} = \frac{5\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z}{3\sqrt{3}} \,.$$

7 Calculate the area of parallelograms formed by the vectors $\mathbf{a}, \mathbf{b}$ if
a) $\mathbf{a} = 2\mathbf{e} + 2\mathbf{f} + 3\mathbf{g}, \mathbf{b} = \mathbf{e} + \mathbf{f} + \mathbf{g}$; b) $\mathbf{a} = 5\mathbf{e} + 8\mathbf{f} + 4\mathbf{g}, \mathbf{b} = 10\mathbf{e} + 16\mathbf{f} + 8\mathbf{g}$.

Solution

The area of parallelogram formed by the vectors $\mathbf{a}, \mathbf{b}$ is equal to the length of the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

The length (modulus of vector) does not depend on orientation. Suppose that the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ defines orientation, i.e. $\mathbf{e} \times \mathbf{f} = \mathbf{g}, \mathbf{f} \times \mathbf{g} = \mathbf{e}, \mathbf{g} \times \mathbf{e} = \mathbf{f}$. a) $S = |\mathbf{a} \times \mathbf{b}|$.

$$\mathbf{a} \times \mathbf{b} = (2\mathbf{e} + 2\mathbf{f} + 3\mathbf{g}) \times (\mathbf{e} + \mathbf{f} + \mathbf{g}) = -2\mathbf{g} + 3\mathbf{f} + 2\mathbf{g} - 3\mathbf{e} - 2\mathbf{f} + 2\mathbf{e} = \mathbf{f} - \mathbf{e}$$

$S = |\mathbf{f} - \mathbf{e}| = \sqrt{1+1} = \sqrt{2}$.

b) Vectors $\mathbf{a} = 5\mathbf{e} + 8\mathbf{f} + 4\mathbf{g}$ and $\mathbf{b} = 10\mathbf{e} + 16\mathbf{f} + 8\mathbf{g}$ are collinear (proportional), hence $\mathbf{a} \times \mathbf{b} = 0$, $S = 0$.

8 *In* 2*-dimensional Euclidean space* $\mathbf{E}^2$ *consider vectors* $\mathbf{a} = (3, 2)$, $\mathbf{b} = (7, 5)$, $\mathbf{c} = (17, 12)\,, \mathbf{d} = (41, 29)$. *Calculate areas of the parallelograms* $\Pi(\mathbf{a}, \mathbf{b}), \Pi(\mathbf{b}, \mathbf{c})$ *and* $\Pi(\mathbf{c}, \mathbf{d})$.

Recall that $\mathbf{E}^2$ can be considered as subspace in $\mathbf{E}^3$, and area of parallelogramm formed by two vectors $\mathbf{a}, \mathbf{b}$ is equal to the length of the vector product $\mathbf{a} \times \mathbf{b}$ in $\mathbf{E}^3$ (See lecture notes 1.12.2).

Solution a) $\mathbf{A}(\mathbf{a}, \mathbf{b}) = \left|\det \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\right| = 1$.

b) $\mathbf{A}(\mathbf{b}, \mathbf{c}) = \left|\det \begin{pmatrix} 7 & 5 \\ 17 & 12 \end{pmatrix}\right| = 84 - 85 = -1$.

c) $\mathbf{A}(\mathbf{c}, \mathbf{d}) = \left|\det \begin{pmatrix} 17 & 12 \\ 41 & 29 \end{pmatrix}\right| = 2871 - 2871 = 1$.

$A(\mathbf{x}, \mathbf{y})$ is algebraic area of parallelogram formed by vectors $\mathbf{x}, \mathbf{y}$. It is equal to area $S(\mathbf{x}, \mathbf{y})$ with positive sign if the triple $\{\mathbf{x}, \mathbf{y}, \mathbf{n}\}$ has "left" orientation and it is equal to $-S(\mathbf{x}, \mathbf{y})$ if the triple $\{\mathbf{x}, \mathbf{y}, \mathbf{n}\}$ has "right" orientation.

8a† *Do you see any relations between parallelograms in the exercise above, fractions* $\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$ *and the number...* $\sqrt{2}$*? Can you continue the sequence of these fractions?*

(Hint: Consider the squares of these fractions.)

One can consider continued fraction $\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$, Consider approximations: $a_0 = 1$, $a_1 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_2 = 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}$, and so on we come to the sequence of fractions:

$$a_k = \frac{p_k}{q_k} \text{ where } p_0 = q_0 = 1, q_{k+1} = p_k + q_k, p_k = 2q_k + p_k \,.$$

One can see that $\left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \frac{1}{q_k q_{k+1}}$ which is just another manifestation of the fact that the area of the parallelogram formed by the vectors $\mathbf{a} = (p_k, q_k), \mathbf{b} = p_{k+1}, q_{k+1}$ equals 1. Vectors $\mathbf{a} = (p_k, q_k), \mathbf{b} = p_{k+1}, q_{k+1}$ form the parallelograms which become longer and longer but all have the same area.

9*Let* P *be an operator in* $\mathbf{E}^2$ *such that*

$$\mathbf{a} = P(\mathbf{e}) = 27\mathbf{e} + 40\mathbf{f}\,, \mathbf{b} = P(\mathbf{f}) = -16\mathbf{e} - \frac{71}{3}\mathbf{f}\,.$$

(see problem 5 *in Homework 2.) Compare the areas of parallelograms* $\Pi(\mathbf{e}, \mathbf{f})$, $\Pi(\mathbf{a}, \mathbf{b})$ *and* $\det P$.

Determinant of operator P is equal to 1. (see in detail Homework 2a) to 1. Area of parallelogram $\Pi(\mathbf{e}, \mathbf{f})$ is equal also to 1. Area of parallelogramm $\Pi(\mathbf{a}\,\mathbf{b})$ is also equal to $\Pi(\mathbf{a}\,\mathbf{b}) = |\mathbf{a} \times \mathbf{b}|$. We have that

$$\mathbf{a} \times \mathbf{b} = (27\mathbf{e} + 40\mathbf{f}) \times (-16\mathbf{e} - \frac{71}{3}\mathbf{f}) = \left(27 \cdot \left(-\frac{71}{3}\right) - 40 \cdot (-16)\right) \mathbf{g} = (-9 \cdot 71 + 640)\,\mathbf{n} = \det P\mathbf{n}\,.$$

We see that determinant of operator is equal to the ratio of areas of parallelograms.

10 *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *be three vectors in* $\mathbf{E}^3$ *such that* $\mathbf{x} = \mathbf{e}+\mathbf{f}, \mathbf{y} = \mathbf{e}+5\mathbf{f}+\mathbf{g}$ *and* $\mathbf{z} = 2\mathbf{f}+3\mathbf{g}$*, where* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is an orthonormal basis in* $\mathbf{E}^3$*.*

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be three another vectors such that*

$$\mathbf{a} = 2\mathbf{x} + 5\mathbf{y} + 7\mathbf{z} \quad \mathbf{b} = \mathbf{x} + 3\mathbf{y} + 2\mathbf{z}\,, \quad \mathbf{c} = 2\mathbf{z}\,. \qquad (*)$$

Find volume of the parallellepiped $\Pi(\mathbf{x}, \mathbf{y}\,\mathbf{z})$

Find volume of the parallellepiped $\Pi(\mathbf{a}, \mathbf{b}\,\mathbf{c})$

Relation (*) defines linear operator.

Calculate determinant of this linear operator.

Volume of parallelepiped $\Pi(\mathbf{x}, \mathbf{y}\,z)$ is equal to

$$|Vol(\Pi(\mathbf{x}, \mathbf{y}, \mathbf{z}))| = |(\mathbf{x}, (\mathbf{y} \times \mathbf{z}))| = \left|\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & 3 \end{pmatrix}\right| = 10\,.$$

(Here we calculate the absolute value, modulus of product $(\mathbf{x}, \mathbf{y} \times \mathbf{z})$. Determinant is positive, hence it is equal to its modulus.) Calculate the area of parallelepiped $\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ straightforwardly: We have

$$\begin{cases} \mathbf{a} = 2\mathbf{x} + 5\mathbf{y} + 7\mathbf{z} = 2(\mathbf{e}+\mathbf{f}) + 5(\mathbf{e}+5\mathbf{f}+\mathbf{g}) + 7(2\mathbf{f}+3\mathbf{g}) = 7\mathbf{e} + 41\mathbf{f} + 26\mathbf{g} \\ \mathbf{b} = \mathbf{x} + 3\mathbf{y} + 2\mathbf{z} = \mathbf{e}+\mathbf{f} + 3(\mathbf{e}+5\mathbf{f}+\mathbf{g}) + 2(2\mathbf{f}+3\mathbf{g}) = 4\mathbf{e} + 20\mathbf{f} + 9\mathbf{g} \\ \mathbf{c} = 2\mathbf{z} = 4\mathbf{f} + 6\mathbf{g} \end{cases}$$

Hence

$$Vol(\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})) = |(\mathbf{a}, (\mathbf{b} \times \mathbf{c}))| = \left|\det \begin{pmatrix} 7 & 41 & 26 \\ 4 & 20 & 9 \\ 0 & 4 & 6 \end{pmatrix}\right| = 20\,.$$

This is litte bit long calculatios...

Now do it in another way:

Vectors $\mathbf{x}, \mathbf{y}\,z$ form basis, since volume of the paralelepiped formed by these vectors is not vanished. Relations (*) defines an operator, which transofrms basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ to triple of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$: $P(\mathbf{x}) = \mathbf{a}\,, P(\mathbf{y}) = \mathbf{b}, P(\mathbf{z}) = \mathbf{c}$. Matrix of this operator is

$$\begin{pmatrix} 2 & 1 & 0 \\ 5 & 3 & 0 \\ 7 & 2 & 2 \end{pmatrix},$$

and for an arbitrary vector

$$\mathbf{A} = p\mathbf{x} + q\mathbf{y} + r\mathbf{z} \Rightarrow P(\mathbf{A}) = pP(\mathbf{x}) + qP(\mathbf{y}) + rP(\mathbf{z})\,,$$

We see that determinant of this operator is equal to $2 \cdot (2 \cdot 3 - 5 \cdot 1) = 2$. Hence $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is also a basis and volume of parallelepiped $\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is twoice more than the volume of parallelepiped $\Pi(\mathbf{x}, \mathbf{y}\,z)$: $Vol(\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})) = 2 \cdot Vol(\Pi(\mathbf{x}, \mathbf{y}\,\mathbf{z})) = 2 \cdot 10 = 20$ ∎