Homework 5b. Solutions.

Here we focus attention on calculations in Cartesian and polar coordinates

1 *Calculate differential forms* $\omega = xdy - ydx$, $\sigma = xdx + ydy$ *and vector fields* $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$ *in polar coordinates.*

This exercise was done during the XII-th lecture (see the subsection 2.3.5 "Differential forms in arbitrary coordinates" in Lecture notes). Just recall the answers:

$$\omega = xdy - ydx = r^2 d\varphi\,, \sigma = xdx + ydy = rdr\,, \quad \mathbf{A} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r}\,, \mathbf{B} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \varphi}\,, \tag{1.1}$$

where

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases}. \tag{1.2}$$

2 *Consider differential forms* $\omega = xdy - ydx$, $\sigma = xdx + ydy$ *and vector fields* $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$

Calculate $\omega(\mathbf{A}), \omega(\mathbf{B})$, $\sigma(\mathbf{A}), \sigma(\mathbf{B})$.

We will solve this problem first in Cartesian coordinates then in polar coordinates

Cartesian coordinates

:

$\omega(\mathbf{A}) = (xdy - ydx)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) =$

$$x^2 dy\left(\frac{\partial}{\partial x}\right) + xydy\left(\frac{\partial}{\partial y}\right) - yxdx\left(\frac{\partial}{\partial x}\right) - y^2 dx\left(\frac{\partial}{\partial y}\right) = x^2\cdot 0 + xy\cdot 1 - yx\cdot 1 - y^2\cdot 0 = 0\,.$$

Later we often denote vector field $\frac{\partial}{\partial x}$ by ∂_x, vector field $\frac{\partial}{\partial y}$ by ∂_y...

$\omega(\mathbf{B}) = (xdy - ydx)(x\partial_y - y\partial_x) = x^2dy(\partial_y) - xydy(\partial_x) - yxdx(\partial_y) + y^2dx(\partial_x) = x^2\cdot 1 - xy\cdot 0 - yx\cdot 0 + y^2\cdot 1 = x^2 + y^2 = r^2$,

$\sigma(\mathbf{A}) = (xdx + ydy)(x\partial_x + y\partial_y) = x^2dx(\partial_x) + xydx(\partial_y) + yxdy(\partial_x) + y^2dy(\partial_y) = x^2\cdot 1 + xy\cdot 0 + yx\cdot 0 + y^2\cdot 1 = x^2 + y^2 = r^2$,

$\sigma(\mathbf{B}) = (xdx + ydy)(x\partial_y - y\partial_x) = x^2dx(\partial_y) - xydx(\partial_x) + yxdy(\partial_y) - y^2dy(\partial_x) = x^2\cdot 0 - xy\cdot 1 + yx\cdot 1 - y^2\cdot 0 = 0$.

Polar coordinates

Using formulae (1.1) and (1.2) from previous problem we come to

$$\omega(\mathbf{A}) = r^2 d\varphi\left(r\frac{\partial}{\partial r}\right) = 0 \ \ ,\omega(\mathbf{B}) = r^2 d\varphi\left(\frac{\partial}{\partial \varphi}\right) = r^2 = x^2+y^2 \ \ ,\sigma(\mathbf{A}) = rdr\left(r\frac{\partial}{\partial r}\right) = r^2 = x^2+y^2 \ \ ,$$

`Note that for this exercise, the solution in polar coordinates is much more shorter!`

3 *Consider a function* $f = x^3 - y^3$.

Calculate the value of 1*-form* $\omega = df$ *on the vector field* $\mathbf{B} = x\partial_y - y\partial_x$.

`In Cartesian coordinates`

$$df(\mathbf{B}) = \partial_{\mathbf{B}} f = (x\partial_y - y\partial_x)(x^3 - y^3) = -3xy^2 - 3yx^4 = --3xy(x+y)\,.$$

Another solution in Cartesian coordinates $\omega = df = 3x^2dx - 3y^2dy$, thus

$$\omega(\mathbf{B}) = 3x^2dx - 3y^2dy(x\partial_y - y\partial_x) = -3x^2ydx(\partial_x) - 3y^2dy(\partial_y) = -3xy(x+y)\,.$$

in polar coordinates:

$f = x^3 - y^3 = r^3(\cos^3\varphi - \sin^3\varphi)$ hence using (1.1) we come to

$$df(\mathbf{B}) = \partial_{\mathbf{B}} f = \frac{\partial}{\partial\varphi}\left(r^3(\cos^3\varphi - \sin^3\varphi)\right) = -3r^3\sin\varphi\cos\varphi(\cos\varphi + \sin\varphi) = -3xy(x+y)\,.$$

4 *Calculate the derivatives of the functions* $f = x^2+y^2$, $g = y^2-x^2$ *and* $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ *(q is a constant) along vector fields* $\mathbf{A} = x\partial_x + y\partial_y$ *and* $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

First do using directional derivatives in Cartesian coordinates, then using formula (1.1), (1.2) in polar coordinates:

For vector field $\mathbf{A} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r}$, we have

`in Cartesian coordinates` $\partial_{\mathbf{A}} f = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x^2+y^2) = x\cdot 2x + y\cdot 2y = 2(x^2+y^2)$,

`in polar coordinates`: $f = r^2$, $\partial_{\mathbf{A}} f = r\frac{\partial}{\partial r}r^2 = 2r^2 = 2(x^2+y^2)$.

`in Cartesian coordinates` $\partial_{\mathbf{A}} g = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(y^2-x^2) = x\cdot(-2x) + y\cdot 2y = 2(y^2-x^2)$,

`in polar coordinates`: $g = r^2(\sin^2\varphi - \cos^2\varphi) = -2r^2\cos 2\varphi$, $\partial_{\mathbf{A}} g = r\frac{\partial}{\partial r}(-r^2\cos 2\varphi) = 2r^2 = -2r^2\cos 2\varphi$.

`in Cartesian coordinates` $\partial_{\mathbf{A}} h = x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = \frac{x^2q}{x^2+y^2} + \frac{y^2q}{x^2+y^2} = q$,

`in polar coordinates`: $\partial_{\mathbf{A}} h = r\frac{\partial}{\partial r}q\log r = q$.

For vector field $\mathbf{B} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial\varphi}$,

we have

`in Cartesian coordinates` $\partial_{\mathbf{B}} f = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(x^2+y^2) = -y\cdot 2x + x\cdot 2y = 0$,

`in polar coordinates`: $f = r^2$, $\partial_{\mathbf{B}} f = \frac{\partial}{\partial\varphi}r^2 = 0$.

`in Cartesian coordinates` $\partial_{\mathbf{B}} g = -y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = -y\cdot(-2x) + x\cdot 2y = 4xy$, `in polar coordinates`: $g = -r^2\cos 2\varphi$, $\partial_B g = \frac{\partial}{\partial\varphi}(-r^2\cos 2\varphi) = 2r^2\sin 2\varphi = 4r^2\cos\varphi\sin\varphi = (4r\cos\varphi)(r\sin\varphi) = 4xy$.

`in Cartesian coordinates` $\partial_{\mathbf{B}} h = -y\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$,
`in polar coordinates`: $\partial_{\mathbf{B}} h = \frac{\partial}{\partial\varphi} q\log r = 0$.

b) Now calculate using 1-form using the fact that $\partial_{\mathbf{A}} f = df(\mathbf{A})$:

We have that $df = d(x^2+y^2) = 2xdx + 2ydy$, $dg = d\left(y^2 - x^2\right) = g_x dx + g_y dy = (2ydy - 2xdx)$, $dh = d\left(q\log\sqrt{x^2+y^2}\right) = h_x dx + h_y dy = \frac{qxdx+qydy}{x^2+y^2}$.

Hence

`in Cartesian`: $\partial_{\mathbf{A}} f = df(\mathbf{A}) = (2xdx+2ydy)(x\partial_x + y\partial_y) = 2x^2dx(\partial_x) + 2y^2dy(\partial_y) = 2x^2+2y^2$,
`in polar` $f = r^2$, $= df = 2rdr$, $\partial_{\mathbf{A}} f = df(\mathbf{A}) = (2rdr)(r\partial_r) = 2r^2$.

`in Cartesian:` $\partial_{\mathbf{A}} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_x + y\partial_y)) = 2ydy(y\partial_y) - 2xdx(x\partial_x) = 2y^2 - 2x^2$.
`in polar` $g = -r^2\cos 2\varphi$, $dg = 2r^2\sin 2\varphi d\varphi - 2r\cos 2\varphi dr$, $\partial_{\mathbf{A}} g = dg(\mathbf{A}) = (2r^2\sin 2\varphi d\varphi - 2r\cos 2\varphi dr)(r\partial_r) = -2r^2\cos 2\varphi$.

in Cartesian $\partial_{\mathbf{A}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2}(x\partial_x + y\partial_y) = \frac{qxdx(x\partial_x)+qydy(y\partial_y)}{x^2+y^2} = \frac{qx^2+qy^2}{x^2+y^2} = q$
in polar $h = q\log r$ $\partial_{\mathbf{A}} h = dh(\mathbf{A}) = \frac{qdr}{r}\left(\frac{\partial}{\partial r}\right) = q$.

Now for vector field $\mathbf{B}$
`in Cartesian` $\partial_{\mathbf{B}} f = df(\mathbf{B}) = (2xdx+2ydy)(-y\partial_x + x\partial_y) = -2xydx(\partial_x) + 2xydy(\partial_y) = 0$,
`in polar`, $f = r^2$, $df = 2rdr$, $\partial_{\mathbf{B}} f = df(\mathbf{B}) = (2rdr)(\partial_\varphi) = 0$,

`in Cartesian` $\partial_{\mathbf{B}} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_y - y\partial_x)) = 2ydy(x\partial_y) - 2xdx(-y\partial_x) = 2xy + 2xy = 4xy$.
`in polar`, $g = -r^2\cos 2\varphi$, $\partial_{\mathbf{B}} g = dg(\mathbf{B}) = (-2r\cos 2\varphi dr + 2r^2\sin 2\varphi d\varphi)\left(\frac{\partial}{\partial\varphi}\right) = 2r^2 sin 2\varphi = 4r^2\sin\varphi\cos\varphi = 4xy$.

`in Cartesian` $\partial_{\mathbf{B}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2}(-y\partial_x + x\partial_y) = \frac{qxdx(-y\partial_x)+qydy(x\partial_y)}{x^2+y^2} = \frac{-qxy+qxy}{x^2+y^2} = 0$.
`in polar` $h = q\log r$, $dh = \frac{qdr}{r}$, $\partial_{\mathbf{B}} h = dh(\mathbf{B}) = \frac{qdr}{r}\left(\frac{\partial}{\partial\varphi}\right) = 0$.

5 *Let f be a function on $\mathbf{E}^2$ given by $f(r,\varphi) = r^3\cos 3\varphi$, where r,φ are polar coordinates in $\mathbf{E}^2$.*

Calculate the 1-form $\omega = df$.

Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r\partial_r + \partial_\varphi$.

Express the 1-form ω in Cartesian coordinates x, y[1)]

[1)] *You may use the fact that $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$.*

$\omega = 3r^2 cos3\varphi dr - 3r^3 \sin 3\varphi d\varphi$.

The value of the form $\omega = df$ on the vector field $\mathbf{X} = r\partial_r + \partial_\varphi$ is equal to

$$\omega(\mathbf{A}) = \left(3r^2 \cos 3\varphi dr - 3r^2 \sin 3\varphi d\varphi\right)(r\partial_r + \partial_\varphi) = 3r^3 \cos 3\varphi dr(\partial_r) - 3r^3 \sin 3\varphi d\varphi(\partial_\varphi) = 3r^3(\cos 3\varphi -$$

because $dr(\partial_r) = 1, dr(\partial_\varphi) = 0$ and $dr(\partial_\varphi) = 0, d\varphi(\partial_\varphi) = 1$.

Another solution

$$\omega(\mathbf{X}) = df(\mathbf{X}) = \partial_{\mathbf{X}} f = \left(r\frac{\partial}{\partial r} + \frac{\partial}{\partial_\varphi}\right)(r^3 \cos 3\varphi) = r \cdot 3r^2 \cos 3\varphi - 3r^3 \sin 3\varphi = 3r^3(\cos 3\varphi - \sin 3\varphi).$$

To express the form ω in Cartesian coordinates it is easier to express f in Cartesian coordinates and then to calculate $\omega = df$:

$$f = r^3 \cos 3\varphi = r^3(4\cos^3\varphi - 3\cos\varphi) = 4(r\cos\varphi)^3 - 3r^2(r\cos\varphi) = 4x^3 - 3x(x^2+y^2) = x^3 - 3xy^2$$

Hence $\omega = d(x^3 - 3xy^2) = (3x^2 - 3y^2)dx - 5xydy$.

We call 1*-form* ω *exact if there exists a function* F *such that* $\omega = dF$

6 *Show that* 1*-form* $\omega = xdy + ydx$ *is exact.*

Show that 1*-form* $\omega = \sin y dx + x\cos y dy$ *is exact.*

Show that 1*-form* $\omega = x^3 dy$ *is not an exact* 1*=form.*

We have $\omega = xdy + ydx = d(xy)$. Hence this is exact form.

We have $\omega = \sin y dx + x \cos y dy = d(x \sin y)$. Hence this is exact form.

Now show that 1-form $\omega = x^3 dy$ is not an exact 1=form. Suppose it is an exat form. Then there exists a function $F = F(x, y)$ such that

$$\omega = x^3 dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \Rightarrow \begin{cases} 0 = \frac{\partial F}{\partial x} \\ x^3 = \frac{\partial F}{\partial y} \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 F}{\partial y \partial x} = 0 \\ \frac{\partial^2 F}{\partial x \partial y} = 3x^2 \end{cases} \Rightarrow \frac{\partial^2 F}{\partial y \partial x} \neq 0 \frac{\partial^2 F}{\partial x \partial y}, \text{ Contradiction}$$