Solutions of Homework 6

1

Calculate the integrals of the form $\omega = xdy - ydx$ *over the following three curves. Compare answers.*

$$C_1\colon \mathbf{r}(t) \begin{cases} x = R\cos t \\ y = R\sin t \end{cases},\ 0 < t < \pi, \quad C_2\colon \mathbf{r}(t) \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases},\ 0 < t < \frac{\pi}{4}$$

$$\textit{and} \quad C_3\colon \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases},\ -1 \le t \le 1.$$

This question was also discussed in lecture notes.

We have that

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} (xdy - ydx)(x_t\partial_x + y_t\partial_y)dt = \int_{t_1}^{t_2} (-y(t)x_t(t) + x(t)y_t(t))dt\,,$$

where $\mathbf{v} = (x_t, y_t)$ is velocity vector: $dx(\partial_x) = dy(\partial_y) = 1$, $dx(\partial_y) = dy(\partial_x) = 0$.)

For the first curve C_1 we have $\mathbf{v}(t) = (-R\sin t, R\cos t)$ and $\int_{C_1}\omega = \int_0^\pi (xdy - ydx)(-R\sin t\partial_x + R\cos t\partial_y) =$

$$\int_0^\pi (R\cos tdy - R\sin tdx)(-R\sin t\partial_x + R\cos t\partial_y) = \int_0^\pi (R^2\cos^2 t + R^2\sin^2 t)dt = \int_0^\pi R^2\cdot dt = \pi R^2\,.$$

For the second curve C_2 we have $\mathbf{v}(t) = (-4R\sin 4t, 4R\cos 4t)$ and $\int_{C_2}\omega = \int_0^{\frac{\pi}{4}} (xdy - ydx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) =$

$$\int_0^{\frac{\pi}{4}} (R\cos 4tdy - R\sin 4tdx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) = \int_0^{\frac{\pi}{4}} (4R^2\cos^2 4t + 4R^2\sin^2 4t)dt = \int_0^{\frac{\pi}{4}} 4R^2\cdot$$

Answer is the same. The second curve is reparameterised first curve ($t \mapsto 4t$) and reparameterisation preserves the orientation: $(4t)' = 4 > 0$.

For the third curve C_3 we have $\mathbf{v}(t) = \left(-R, -\frac{Rt}{\sqrt{1-t^2}}\right)$ and $\omega(\mathbf{v}(t)) = (xdy - ydx)(v_x\partial_x + v + y\partial_y) =$

$$= \left(Rtdy - R\sqrt{1-t^2}dx\right)\left(R\partial_x - \frac{Rt}{\sqrt{1-t^2}}\partial_y\right) = -R^2\sqrt{1-t^2} - \frac{R^2t^2}{\sqrt{1-t^2}} = -\frac{R^2}{1-t^2}\,.$$

Hence

$$\int_{C_3}\omega = \int_0^1 \omega(\mathbf{v}(t)) = \int_0^1 \left(-\frac{R^2}{\sqrt{1-t^2}}\right)dt = -R^2\int_0^1 \frac{dt}{\sqrt{1-t^2}} = -\pi R^2\,.$$

Answer is the same up to a sign: This curve is reparameterised first curve. If we put $t = \cos\tau$ then third curve C_3 will transform to the first curve C_1. This reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples was considered an integral over the same (non-parameteresed) half-circle. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation. Sure if we already know the information about orientation of these curves we did not need to calcualte all the three integrals: $\int_{C_1}\omega$,$\int_{C_2}\omega$, and $\int_{C_3}\omega$. It is enough to calculate one of these integrals, e.g. the first one (the calculations are little bit simpler for first one) then use the fact that curve C_2 has the same orientation with curve C_1 and curve C_3 has opposite orientation to the curve C_1, hence

$$\int_{C_1}\omega = \pi R^2 = \int_{C_2}\omega = -\int_{C_3}\omega\,.$$

(For solutions see also lecture notes the end of subsection 2.5)

2

Consider an arc of parabola $x = 2y^2 - 1$, $0 < y < 1$.

Give examples of two different parameterisations of this curve such that these parameterisations have the opposite orientation.

Calculate the integral of the form 1-form $\omega = \sin y dx$ over this curve.

How does the answer depend on a parameterisation?

To consider a different parameterisation we may take an arbitrary number $n \neq 0$ and consider

$$C_n\colon \mathbf{r}(t)\ \begin{cases} x = 2n^2t^2 - 1 \\ y = nt \end{cases},\ 0 < t < 1/n,$$

These two different parameterisation are related with the reparameterisation $t' = nt$. If $n > 0$, then reparameterisation preserves orientation, If $n < 0$, then reparameterisation changes orientation of the curve. For example if we take $n = 2$ then we will come to the curve

$$C_2\colon \mathbf{r}(t)\ \begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases},\ 0 < t < 1/2,$$, with the same orientation as initial curve

and if we will take $n = -2$ we will come to the curve

$$C_2'\colon \mathbf{r}(t)\ \begin{cases} x = 8t^2 - 1 \\ y = -2t \end{cases},\ -\tfrac{1}{2} < t < 0,$$,

we will come to the curve with orientation opposite to the orientation opposite to the orientation of the initial curve.

Sure we can change parametersiation in a different way. E.g. we may consider

$$C_3\colon \mathbf{r}(t)\ \begin{cases} x = 2\cos^2 t - 1 = \cos 2t \\ y = \cos t \end{cases},\ 0 < t < \tfrac{\pi}{2}$$

Curve C_3 has orientation opposite to the orientation of the curves C, C_2 and the same orienation with the curve C_2' since reparameterisation $t' = \cos t$ changes orienation ($\frac{dt'}{dt} = -\sin t < 0$ for $0 \leq t \leq \frac{\pi}{2}$).

Now we calculate integrals for all these curves. Sure we do not need to do it, it suffices to calculate the integral just for one curve, and then using orientation arguments to find integrals for other curves, but just for exercise we will do all examples.

For any curve $\mathbf{r}(t), t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t) \frac{dx(t)}{dt} dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve C_1 $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t \sin t dt = 4(-t\cos t + \sin t)\Big|_0^1 = -4\cos 1 + 4\sin 1$$

For the second curve C_2 $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t \sin 2t dt = 4(-2t\cos 2t + \sin 2t)\Big|_0^{1/2} = -4\cos 1 + 4\sin 1\,.$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve C_2' $x_t = 16t$ and

$$\int_{C_2} \omega = \int_{-1/2}^{0} 16t \sin(-2t) dt = -4(-2t\cos 2t + \sin 2t)\Big|_{-1/2}^{0} = 4\cos 1 - 4\sin 1\,.$$

Answer is the same up to a sign. Non-surprising. This curve is reparameterised first curve ($t \mapsto -2t$) and reparameterisation changes the orientation.

For the last curve $x_t = -2\sin 2t dt$ and

$$\int_{C_3} w = \int_0^{\frac{\pi}{2}} (-2\sin 2t)\sin(\cos t) dt = -4\left(\cos t\cos(\cos t) - \sin(\cos t)\right)\Big|_{-1/2}^{\pi/2} = 4\cos 1 - 4\sin 1$$

Answer is the same as for the previous curve: This curve is reparameterised first curve with opposite orientation ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$. hence the first integral is equal to the third on and it has a sign opposite to the second and first one.

Resumé: In these three examples an integral over the same (non-parameteresed) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

3

Calculate the integral of the form $\omega = xdy$ over the following curves
a) closed curve $x^2 + y^2 = 12y$.
b) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

a) Consider closed curve $x^2 + y^2 = 12y$. We have

$$0 = x^2 + y^2 - 12y = x^2 + (y-6)^2 - 36\,.$$

That is this curve is a circle of the radius 6 with a centre at the point $(0, 6)$. The parametric equation of this circle is

$$\begin{cases} x = 6\cos t \\ y = 6 + 6\sin t \end{cases}, \quad 0 \leq t \leq 2\pi\,.$$

We have that

$$\mathbf{v} = \begin{pmatrix} -6\sin t \\ 6\cos t \end{pmatrix} \quad \text{and} \quad \omega(\mathbf{v}) = xdy(v_x\partial_x + v_y\partial_y) = xv_y = 6x(t)\cdot 6\cos t = 36\cos^2 t\,,$$

$$\int_C \omega = \int_0^{2\pi} \omega(\mathbf{v}(t))dt = \int_0^{2\pi} 36\cos^2 t\,dt = 36\cdot\frac{2\pi}{2} = 36\pi\,.$$

So for an arbitrary parameterisation answer will be $\pm 36\pi$. (36π if orientation is the same and -36π if opposite) E.g. if we change parameterisation above on the parameterisation $\tau = -t$ then integral will change a sign, since this reparameterisation changes the orientation of the circle.

b) For the the arc of the ellipse $x^2 + y^2/9 = 1, y \geq 0$ choose a parameterisation: $\begin{cases} x = \cos t \\ y = 3\sin t \end{cases}$, $0 \leq t \leq \pi$. Then $\mathbf{v} = (-\sin t, 3\cos t)$ and

$$\int_C \omega = \int_0^{\pi} \omega(\mathbf{v})dt = \int_0^{\pi} x(t)y_t dt = \int_0^{\pi 2} 3\cos t\cos t dt = \int_0^{\pi} 3\cos^2 t dt = 3\pi/2$$

So for an arbitrary parameterisation answer will be $\pm 3\pi/2$, sign is depending on orientation of parameterisation. E.g. if we change parameterisation above on the parameterisation $\tau = -t$ then integral will change a sign, since this reparameterisation changes the orientation of the ellipse.

4 *a) Calculate the integrals $\int_{C_1}\omega$ and $\int_{C_2}\omega$ of the 1-form $\omega = xdy - ydx$ over the curves C_1: $x^2 + y^2 = 9$ and C_2: $x^2 + y^2 = 6y$.*

b) Perform the calculations of integrals $\int_{C_1}\omega$ and $\int_{C_2}\omega$ in polar coordinates.

Hint Performing the calculations for the curve C_2 one may use the polar coordinates r', φ' with the centre at the point (a, b): $\begin{cases} x = a + r\cos\varphi \\ y = b + r\sin\varphi \end{cases}$.

These both curves are circles, in particular C_2: $x^2 + y^2 = 6y \Leftrightarrow (y-3)^2 + x^2 = 9$

Choose for these curves parameterisations:

$$C_1\colon \begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}, 0 \le t < 2\pi\,, \qquad \begin{cases} x = 3\cos t \\ y = 3 + 3\sin t \end{cases}, 0 \le t < 2\pi\,. \tag{4.1}$$

For the both curves the velocity vector $\mathbf{v} = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -3\sin t \\ 3\cos t \end{pmatrix}$ is the same.

We have that for the first curve

$$\omega(\mathbf{v}(t)) = xdy - ydx\left(-3\sin t\frac{\partial}{\partial x} + 3\cos t\frac{\partial}{\partial y}\right) = 3x(t)\cos t + 3y(t)\sin t = 9\cos^2 t + 9\sin^2 t = 9\,,$$

hence

$$\int_{C_1} \omega = \int_0^{2\pi} \omega(\mathbf{v}(t))dt = \int_0^{2\pi} 9dt = 18\pi\,.$$

Now perform analogous calculations for the second curve C_2:

$$\omega(\mathbf{v}(t)) = xdy - ydx\left(-3\sin t\frac{\partial}{\partial x} + 3\cos t\frac{\partial}{\partial y}\right) = 3x(t)\cos t + 3y(t)\sin t = 3(3\cos t)\cos t + (3 + 3\sin t)3\,\mathrm{s}$$

hence

$$\int_{C_1} \omega = \int_0^{2\pi} \omega(\mathbf{v}(t))dt = \int_0^{2\pi} (9 + 9\sin t)dt = 18\pi\,.$$

The answer is the same (it is the area of the interior of the circle.).

`Now perform calcualtions in polar coordinates.`

Recall that the form $\omega = xdy - ydx = d\varphi$ in polar coordinates (see lecture notes) is equal to $r^2 d\varphi$:

$$xdy - ydx = (r\cos\varphi)(\sin\varphi dr + r\cos\varphi d\varphi) - (r\sin\varphi)(\cos\varphi dr - r\sin\varphi d\varphi) = r^2(\cos^2\varphi + \sin^2\varphi)d\varphi = r^2 d\varphi$$

The equation of the curve C_1 in polar coordinates is $\begin{cases} r(t) = 3 \\ \varphi(t) = t \end{cases}, 0 \le t < 2\pi$. Velocity vecor $\mathbf{v} = \begin{pmatrix} r_t \\ \varphi_t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\omega(\mathbf{v}(t)) = r^2(t)d\varphi\left(\frac{\partial}{\partial\varphi}\right) = 9$, and

$$\int_{C_1} \omega = \int_0^{2\pi} \omega(\mathbf{v}(t)) = 0 \cdot 2\pi = 18\pi.$$

In the case of the second curve we have to choose coordinates, adjusted to this curve. Choose polar coordinates (u, θ) with the centre at the point $(0, 3)$ such that

$$\begin{cases} x = u\cos\theta \\ y = 3 + u\sin\theta \end{cases}.$$

Then in these polar coordinates the curve C_2: $x^2 + y^2 = 6y$ looks nice: one can see that the equation of the curve C_2 in polar coordinates u, θ is $\begin{cases} u(t) = 3 \\ \theta(t) = t \end{cases}, 0 \leq t < 2\pi$. C_2 then we have

$$xdy - ydx = u\cos\theta((\sin\theta du + u\cos\theta d\theta) - (3 + u\sin\theta)(\cos\theta du - u\sin\theta d\theta) = u^2(\cos^2\theta + \sin^2\theta)d\theta =$$

$$u^2 d\theta - 3\cos\theta du + 3u\sin\theta d\theta\,,$$

and $\omega(\mathbf{v}(t)) = u^2 d\theta - 3\cos\theta du + 3u\sin\theta d\theta\left(\frac{\partial}{\partial\theta}\right) = 9 + 9\sin t$,

the appearance of the form ω and $\omega(\mathbf{v}(t))$ is not such simple as for the first curve, but the result of integration is the same:

$$\int_{C_1} \omega = \int_0^{2\pi} \omega(\mathbf{v}(t)) = \int_0^{2\pi} (9 + 9\sin t)dt = 18\pi.$$

5

Calculate the integral $\int_C \omega$ *where* $\omega = xdx + ydy$ *and* C *is*

a) the straight line segment $x = t, y = 1 - t$, $0 \leq t \leq 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \leq t \leq 1$, $n = 2, 3, 4, \ldots$

c) **an arbitrary** *curve starting at the point* $(0, 1)$ *and ending at the point* $((1, 0)$.

For any of these curves we can perform calculations naively just using definition of integral

E.g. for the curve a)

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1 - t)(-1))dt = \int_0^1 (2t - 1)dt = 0\,,$$

for the curve b) if $n = 2$

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1 - t^2)(-2t))dt = \int_0^1 (2t^3 - 3t^2)dt = 0\,,$$

for the curve b) in general case:

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt =$$

$$\int_0^1 (t + (1 - t^n)(-nt^{n-1}))dt = \int_0^1 (t - nt^{n-1} + nt^{2n-1})dt = 0\,.$$

But there is another nice way to calculate these integrals. We immediately come to these results in a clear and elegant way if we use the fact that $\omega = xdx + ydy$ is an **exact**

form, i.e. $\omega = df$ where $f = \frac{x^2+y^2}{2}$. Indeed using Theorem we see that for an arbitrary curve starting at the point $A = (0, 1)$ and ending at the point $B = (1, 0)$

$$\int_C \omega = \int_C df = f(x, y)|_A^B = f(1, 0) - f(0, 1) = 0\,.$$

6

Show that the form 1-form $\omega = 3x^2ydx + x^3dy$ *is an exact 1-form. Calculate integral of this form over the curves considered in exercises 2) and 3)*

One can see that $\omega = 3xydx + x^3dy = d(x^3y)$ $(d(x^3y) = \frac{\partial(x^3y)}{\partial x}dx + \frac{\partial(x^3y)}{\partial y}dy = 3x^2ydx + x^3dy.)$

It is an exact form.

Integral of this exact form over the circle $x^2 + y^2 = 12y$ (exercise 2a) equals to zero, since it is closed curve: starting and ending points coincide.

Integral of this exact form over the arc of the ellipse $x^2 + y^/9 = 1(exercise 2b), y \geq 0$ and the integral over arc of the unit circle $x^2 + y^2 = 1, y > 0$ both are equal zero in spite of the fact that these curves are not closed. The reason is that the function $f = x^3y$ $(df = \omega)$ vanishes at starting and ending points of these curves.

The integral of this form over arc of the unit circle starting at the point $A = (4, 0)$ and ending at the point $(2, 0)$ (see the exercise 3) is equal to $\int_C \omega = f|_B^A = f(1, 0) = f(0, 1) = 0$ because $f = x^2y$ and $f(1, 0) = f(0, 1) = 0$. Answer is equal to zero. Hence it does not depend on orientation of the curve.

7.

Consider the following differential 1-forms in $\mathbf{E}^2$*:*

a) xdx *, b)* xdy *c)* $xdx + ydy$ *, d)*$xdy + ydx$*, e)* $xdy - ydx$

f) $x^4dy + 4x^3ydx$

a) Show that 1-forms a), c), d) and f) are exact forms

b)Why 1-forms b) and e) are not exact?

a) It is an exact form since $xdx = df$ where $f = \frac{x^2}{2} + c$, where c is a constant.

b) Suppose $\omega = xdy$ is an exact form: $\omega = df = f_xdx + f_ydy$. Hence $f_x = 0, f_y = x$. We see that $f_{xy} = \frac{\partial}{\partial x}f_y = 1$. On the other hand $f_{yx} = \frac{\partial}{\partial y}f_x = f_{xy} = 0$. Contradiction.

Another solution; There is another way to show why $\omega = xdy$ is not an exact form. We already calculated that the integral of the form $\omega = xdy$ over the closed circle $x^2+y^2 = 12y$ is equal to $36\pi \neq 0$. (see the exercise 4) and its solution above)Hence ω is not exact, since the integral of an exact form over an arbitrary closed curve is equal to zero.

c) It is an exact form since $xdx + ydy = d\left(\frac{x^2+y^2}{2} + c\right)$, ($c$ is a constant).

d)It is an exact form since $xdy + ydx = d(xy + c)$,where c is a constant.

e)Suppose $\omega = xdy - ydx$ is an exact form: $\omega = df = f_x dx + f_y dy$. Hence $f_x = -y, f_y = x$. We see that $f_{xy} = 1$. On the other hand $f_{yx} = f_{xy} = -1$. Contradiction.

f) It is an exact form since $x^4dy + 4x^3ydx = d(x^4y + c)$, where c is a constant.

8 *Consider* 1*-form*

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$*, i.e. in all the points except origin:* $x^2 + y^2 \neq 0$.

a) Write down this form in polar coordinates

b) † *What values can take the integral* $\int_C \omega$ *of this form, if* C *is an arbitrary curve starting at the point* $(0, 1)$ *and ending at the point* $((1, 0)$ *(we suppose that the curve* C *does not pass trough the origin)*

a) We know (see the exercise 4 or Lecture notes) that in polar coordinates $xdy - ydx = r^2 d\varphi$, hence

$$\omega = \frac{xdy - ydx}{x^2 + y^2} = d\varphi \,.$$

b) † if the curve **does not pass the origin** then the integral is well-defined, It is equal $\frac{\pi}{2} + 2\pi n$. The integer n depends on the curve.

9† *Let* $\omega = a(x, y)dx + b(x, y)dy$ *be a closed form in* $\mathbf{E}^2$*,* $d\omega = 0$.

Consider the function

$$f(x, y) = x\int_0^1 a(tx, ty)dt + y\int_0^1 b(tx, ty)dt \tag{2}$$

† *Show that*

$$\omega = df \,.$$

(This proves that an arbitrary closed form in $\mathbf{E}^2$ *is an exact form.*

† *Why we cannot apply the formula (2) to the form* ω *defined by the expression (1)?*

Perform the calculations: $df = f_x d + f_y dy$.

$$f_x = \int_0^1 a(tx, ty)dt + x\int_0^1 a_x(tx, ty)tdt + y\int_0^1 b_x(tx, ty)tdt \,.$$

and

$$f_y = \int_0^1 b(tx, ty)dt + x\int_0^1 a_y(tx, ty)tdt + y\int_0^1 b_y(tx, ty)tdt \,.$$

On the other hand $d\omega = d(adx + bdy) = (b_x - a_y)dx \wedge dy = 0$. Hence $b_x = a_y$ and

$$f_x = \int_0^1 a(tx, ty)dt + x\int_0^1 a_x(tx, ty)tdt + y\int_0^1 a_y(tx, ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(ta(tx, ty)\right)\right) = ta(tx, ty)\big|_0^1 = a($$

because

$$\frac{d}{dt}\left(ta(tx,ty)\right) = a(tx,ty) + xta_x(tx,ty) + yta_y(tx,ty).$$

Analogously

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 b_x(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(tb(tx,ty)\right)\right) = tb(tx,ty)\Big|_0^1 = b(x$$

We see that $f_x = a(x,y)$ and $f_y = b(x,y)$, i.e. $df = adx + bdy$ ■