Solutions of Homework 7

1 *Let C be an ellipse in the plane $\mathbf{E}^2$ such that its foci are at points $F_1 = (-1, 0)$ and $F_2 = (1, 0)$ and it passes through the point $K = (0, 2)$.*

Write down the analytical formula which defines this ellipse.

Find the area of this ellipse.

The foic of this ellipse are on the OX axis, and the centre of this ellipse is at the point midpoint of the segment $[F_1F_2]$, it is the origin $L(0, 0)$. Hence the analytical equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a is semi-major axis, and b is semi-minor axis, $a \geq b$. This ellipse intersects the axis OY at the point $x = 0, y = \pm b$. Hence the point $K = (0, 2) = (0, b)$, i.e. semi-minor axis $b = 2$. Calculating the distances between the point K and foci we calculate the length of the semi-major axis

$$|KF_1| + |KF_2| = \sqrt{2^2 + 1} + \sqrt{2^2 + 1} = 2\sqrt{5} = 2a \Rightarrow a = \sqrt{5}$$

We see that $a = \sqrt{5}$ and $b = 2$, thus analytical formual for the ellispe is

$$\frac{x^2}{5} + \frac{y^2}{4} = 1\,.$$

Area of the ellipse is equal to $S = \pi ab = 2\sqrt{5}\pi$.

2 *Let C be an ellipse in the plane $\mathbf{E}^2$ such that its foci are at the points $F_1 = (-5, 0)$, $F_2 = (16, 0)$. It is known that the point $K = (0, 12)$ belongs to the ellipse.*

Find intersections of the ellipse with OX and $0Y$ axis.

Find the area of the ellipse.

We know that $K = (0, 12)$, We have that the point $K = (0, 12)$ belongs to the ellipse. Hence

$$|KF_1| + |KF_2| = \sqrt{12^2 + (-5)^2} + \sqrt{12^2 + 16^2} = \sqrt{169} + 4\sqrt{25} = 33\,.$$

Hence for an arbitrary D point on the ellipse $|DF_1| + |DF_2| = 33$. If the ellispe intersects OX axis at the point with coordinate $(x, 0)$ then

$$|KF_1| + |KF_2| = 33 = |x - (-5)| + |x - 16| = |x + 5| + |x - 16|\,.$$

1-st case: $x > 16$, $|x + 5| + |x - 16| = 2x - 11 = 33 \Rightarrow x = 22$.

2-nd case: $x < -5$, $-x - 5 - x + 16 = -2x + 11 = 33 \Rightarrow x = -11$.

The point of intersectin the OY axis is the point $K' = (0. - 12)$ symmetrical to the point K.

Hence

`The ellipse intersects axis at the points` $(22, 0), (-11, 0), (0, \pm 12)$.

Now find the area of this ellipse.

To calculate area we calculate length of axis of the ellipse, the minor and the major. Foci of this ellipse are on the OX axis. Ellipse intersects OX axis at the points $(-11, 0)$ and $(22, 0)$. Hence major axis is equal to $2a = 33$. The distance between foci $2c = 16 - (-5) = 21$. Hence the minor axis b is equal to

$$b = \sqrt{a^2 - c^2} = \sqrt{\left(\frac{33}{2}\right)^2 - \left(\frac{21}{2}\right)^2} = \frac{1}{2}\sqrt{33^2 - 21^2} = \frac{1}{2}\sqrt{(33-21)(33+21)} =$$

$$\frac{1}{2}\sqrt{12 \cdot 54} = \frac{1}{2}\sqrt{3 \cdot 4 \cdot 3 \cdot 9 \cdot 2} = 9\sqrt{2}\,.$$

One may calculate semi-minor axis b also in another way: The centre is at the midpoint of the segment $(F_1, F_2) = (-5, 16)$: the point $\left(\frac{11}{2}, 0\right)$. If semi-minor axis is equal to b then consider the point $P = \left(\frac{11}{2}, b\right)$. This point is at the equal distances from foci thus $|PF_1| + |PF_2| = 2|PF_1| = 2\sqrt{\left(\frac{11}{2} - (-5)\right)^2 + b^2} = 33$. We come to

$$\left(\frac{11}{2} - (-5)\right)^2 + b^2 = \left(\frac{21}{2}\right)^2 + b^2 \left(\frac{33}{2}\right)^2 \Rightarrow b = \frac{1}{2}\sqrt{33^2 - 21^2} = 9\sqrt{2}\,.$$

We calculated lengths of semi-minor and semi-major axes:

$$a = \frac{33}{2}\,, b = 9\sqrt{2}\,.$$

Hence the area of the ellipse is equal to

$$S = \pi \cdot \text{vertical half-axis} \ \cdot \text{horisontal half-axis} = \pi a b = \frac{297\sqrt{2}}{2}\,.$$

3 3a) *Let H be hyperbola in the plane $\mathbf{E}^2$ such that it passes through the point $P = (2, 3)$, and its foci are at the points $F_{1,2} = (\pm 2, 0)$,*

Find the intersection points of the hyperbola with OX axis.

Write down the analytical formula which defines this hyperbola, i.e. the equation defining this hyperbola in Cartesian coordinates (x, y).

Explain why this hyperbola does not intersect the axis OY.

3*b) Let H be hyperbola in the plane $\mathbf{E}^2$ such that it passes through the point $P = (2, 3)$, and its foci are at the points $F_{1,2} = (0, \pm 2)$,*

Compare this question with the previous one.

Write down the analytical formula which defines this hyperbola.

3a) By geometrical definition of hyperbola we have that for arbitrary point K of hyperbola

$$||KF_1| - |KF_2|| = ||PF_1| - |PF_2|| = |\sqrt{(2-2)^2 + (3-0)^2} - \sqrt{(2-(-2))^2 + (3-0)^2}| = |3 - 5| = 2\,.$$

If the hyperbola intersects OX axis at the point $(x, 0)$ then

$$||KF_1| - |KF_2|| = 2 = ||x-2| - |x - -2|| = ||x-2| - |x+2||\,.$$

This means that $|x-2| - |x+2| = \pm 2$.

1-st case $|x-2| - |x+2| = 2 \Rightarrow x = -1$

2-nd case $|x-2| - |x+2| = -2 \Rightarrow x = 1$

`Hyperbola intersects` OX `axis at the points` $(\pm 1, \pm 0)$.

The anaylitcal formula defining this hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,, \text{with} \quad |F_1F_2| = 2c, b^2 = c^2 - a^2\,.$$

We have $a = \pm 1$, $c = 2$, thus $b^2 = 3$ and equation defining the hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1\,.$$

Check that a point $(2, 3)$ belongs to this hyperbola: $\frac{2^2}{1} - \frac{3^2}{3} = 1$.

This hyperbola does not intersect the axis OY because the points of this axis are on the equal distance from foci of the hyperbola.

3b) In this exercise, the coordinates x and y swap comparing the exercise $3a$)

Hence the equation of this hyperbola is:

$$\frac{y^2}{1} - \frac{x^2}{3} = 1\,.$$

4 *Consider in the plane the curves* C_1*,* C_2 *and* C_3 *which are given in some Cartesian coordinates* (x, y) *by equations* C_1: $4x^2 + 4x + y^2 = 0$, C_2: $4x^2 + 4x - y^2 = 0$,

C_3: $4x^2 + 4x + y = 0$.

Show that C_1 *is ellipse,* C_2 *is hyperbola, and* C_3 *is parabola*

We have

$$C_1\colon 4x^2 = 4x + y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 + y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 + y^2 = 1\,.$$

Choose new Cartesian coordinates $\begin{cases} x + \frac{1}{2} = x' \\ y = y' \end{cases}$ we come to

$$C_1\colon 4x^2 = 4x + y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 + y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 + y^2 = 1 \Leftrightarrow$$

$$y'^2 + 4x'^2 = 1\,.$$

This is canonical equation of ellipse. (x' is the second coordinate, and y' is the first)

Now consider C_2:

$$C_2\colon 4x^2 = 4x - y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 - y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 - y^2 = 1\,.$$

Choose new Cartesian coordinates $\begin{cases} x + \frac{1}{2} = x' \\ y = y' \end{cases}$ we come to

$$C_2\colon 4x^2 = 4x - y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 - y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 - y^2 = 1 \Leftrightarrow$$

$$4x'^2 - y'^2 = 1\,.$$

We see that this is canonical equation of hyperbola

Now consider C_3:

$$C_3\colon 4x^2 + 4x + y = 0\,.$$

This is equation of parabola. To make it canonical we have to choose Cartesian coordinates $\tilde{x}, \tilde{y}$ such that in these coordinates the curve C_3 have the appearance

$$C_3\colon \tilde{y}^2 = 2p\tilde{x}\,.$$

We have

$$\mathbf{C}_3\colon 4x^2 + 4x + y = 0 \leftrightarrow (4x^2 + 4x + 1) + (y - 1) = 0 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 + (y - 1) = 0$$

$$\Leftrightarrow \left(x + \frac{1}{2}\right)^2 = \frac{1}{4}(1 - y) = 0$$

Choose new Cartesian coordinates

$$\begin{cases} \tilde{x} = 1 - y \\ \tilde{y} = x + \frac{1}{2} \end{cases}$$

We see that in these coordinates

$$C_3\colon \tilde{y}^2 = 2p\tilde{x}\,, \quad \text{with } p = \frac{1}{8}\,.$$

5 *Let H be hyperbola considered in the exercise* **3**.

Consider in the plane $\mathbf{E}^2$ the ellpise such that it passes through the foci of the hyperbola H, and its foci are at the points where hyperbola H intersects axis OY. Write down equation of this ellipse.

The foci of this ellipse are on the axis OY and its centre is at the origin, hence the equation of this ellipse is

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1\,, (a > b)$$

The ellipse passes through the foci of the hyperbola the points $(0, \pm 2)$. Hence $\frac{y^2}{a^2} = 1$ for $x = 0$, i.e. $a = 2$. Consider the point $(b, 0)$ on this ellipse. The sum of distances from this point to foci is equal to

$$\sqrt{b^2 + 1^2} + \sqrt{b^2 + (-1)^2} = 2\sqrt{b^2 + 1} = 2a = 4\,. \Rightarrow b = 3\,.$$

Hence the equation of the ellipse is

$$\frac{y^2}{4} + \frac{x^2}{3} = 1\,.$$

Remark Notice that this equation is not canonical one: foci of this ellipse are on the axis OY not on the axis OX. Tis is related with the fact that corresponding hyperbola is not in canoncal coordinates: we have to swap coordinates x, y then and ellipse and hyperbola will be given in canonical Cartesina coordinates.

6 The ellipse C on the plane $\mathbf{E}^2$ has foci at the vertices $A = (-1, -1)$ and $C = (1, 1)$ of the square $ABCD$, and it passes through the other two vertices $B = (-1, 1)$ and $D = (1, -1)$ of this square.

Find new Cartesian coordinates (u, v) (express them via initial coordinates (x, y)) such that the ellipse C has canonical form C: $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ in these coordinates.

Write down the equation of ellipse C in initial Cartesian coordinates (x, y)

Calculate the area of this ellipse.

Consider new Cartesian coordinates $OX'Y'$ such that axis OX' goes from focus $A = (-1, -1)$ to the focus $C = (1, 1)$. In these new Cartesian coordinates equation of the ellipse is

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

The foci are on the axis AC, the semi-minor axis of the ellise: $b = |OB| = \sqrt{2}$, $b = \sqrt{2}$. To calculate a use the fact that $2a$ is equal to the sum of the distances from the arbitrary point to the foci: $2a = |BC| + |BA| = 2|BA| = 4$, and $a = 2$. We see that equation of the ellipse in new Cartesian coordinates is

$$\frac{x'^2}{4} + \frac{y'^2}{2} = 1$$

Now find the relation between new coordinates (x', y') and the genuine coordinates (x, y).

The line AC is the bisectrix of the angle XOY. Hence we come from Cartesian coordinates OXY to Cartesian coordinates $OX'Y'$ by rotation on the angle $\frac{\pi}{4}$: Having in mind that $\cos\frac{\pi}{4} = \frac{\sin\pi}{4} = \frac{\sqrt{2}}{2}$, and that the new coordinates (x', y') of the point A are $(-\sqrt{2}, 0)$, and the new coordinates (x', y') of the point C are $(0, \sqrt{2})$ we see that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{cases} x' = \frac{x+y}{\sqrt{2}} \\ y' = \frac{-x+y}{\sqrt{2}} \end{cases}$$

Thus equation of ellipse is

$$\frac{x'^2}{4} + \frac{y'^2}{2} = 1 \Leftrightarrow \frac{1}{4}\left(\frac{x+y}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{-x+y}{\sqrt{2}}\right)^2 = 1 \Leftrightarrow 3x^2 + 3y^2 - 2xy = 8\,.$$

Area of the ellipse is equal to the product of semi-major axis on semi-minor axis and on π:

$$S = \pi\sqrt{2} \cdot 2 = 2\pi\sqrt{2}\,.$$

7 *Consider a curve defined in Cartesian coordinates (x, y) by the equation*

$$C\colon \quad px^2 + py^2 + 2xy + \sqrt{2}(x + y) = 0\,,$$

where p is a parameter.

How looks this curve

if $p > 1$? if $p = 1$? if $-1 < p < 1$? if $p = -1$? if $p < -1$?

Find an affine transformation

$$\begin{cases} x = au + bv + e \\ y = cu + dv + f \end{cases}, \qquad \left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0\right) \tag{1}$$

which transforms this curve to the circle $u^2 + v^2 = 1$ in the case if $p > 1$

We have

$C\colon px^2 + py^2 + 2xy + \sqrt{2}(x + y) = 0.$

Rotate coordinates on the angle $\frac{\pi}{4}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \begin{cases} x = \frac{u+v}{\sqrt{2}} \\ y = \frac{-u+v}{\sqrt{2}} \end{cases}$$

(Compare with changing of coordinates in the previous exercise).

We have in new Cartesian coordinates (u, v)

$$C: px^2+py^2+2xy+\sqrt{2}(x+y) = p\left(\frac{u+v}{\sqrt{2}}\right)^2+p\left(\frac{-u+v}{\sqrt{2}}\right)^2+2\left(\frac{u+v}{\sqrt{2}}\right)\left(\frac{-u+v}{\sqrt{2}}\right)+2u =$$

$$(p-1)u^2+(p+1)v^2+2u = 0\,.$$

1) $p > 1$

$$C: (p-1)u^2+(p+1)v^2+2u = (p-1)\left(u+\frac{1}{p+1}\right)^2+(p-1)v^2 = \frac{1}{p-1}$$

This is an ellipse

2) $p = 1$

$$(p-1)u^2+(p+1)v^2+2u = 0 = 2V^2+2U = 0\,.$$

This is a parabola.

3) $-1 < p < 1$

$$C: (p-1)u^2+(p+1)v^2+2u = (p+1)v^2-(1-p)\left(u+\frac{1}{p+1}\right)^2 = \frac{1}{p-1}$$

This is hyperbola.

4) $p = -1$

$$(p-1)u^2+(p+1)v^2+2u = 2(u-u^2) = 2u(u-1)\,.$$

two parallel lines

5) $p < -1$ it is again ellipse:

$$C: (p-1)u^2+(p+1)v^2+2u = (p-1)\left(u+\frac{1}{p+1}\right)^2+(p-1)v^2 = \frac{1}{p-1}$$

it is again an ellipse.

Consider the following transformation from Cartesian coordinates u, v to new coordinates u', v':

$$\begin{cases} u' = \sqrt{p-1}u \\ v' = \sqrt{p+1}v \end{cases}.$$

In these coordinates (they are not Cartesian coordinates!) the curve C for will become a circle.

8* (`pursuit problem`) *Consider two point in the plane* $\mathbf{E}^2$, *A, and B. Let point A starts moving at the origin, and moves along OY with constant velocity v:* $\begin{cases} x = 0 \\ y = vt \end{cases}$.

Let point B starts moving at the point $(L,0)$, its speed is equal to v, and velocity vector is directed on the paricle A, i.e. at any moment of time the particle B moves in the direction of the segment BA with the same speed v.

Of course the particle B never will reach the particle A becuase their speeds are the same. On the other hand the particle B asymptotically will be tended to vertical axis. What is the distance between these particles at $t \to \infty$?

Hint: Consider the reference frame in which particle A is not moved, i.e. consider coordinates $\begin{cases} x' = x \\ y' = y + vt \end{cases}$.

Show that in these coordinates the trajectory of particle B will be a parabola.

See one of the last etudes in my homepage on Geometry:"Cobnic sections and pursuit problem"