Solutions of Homework 8

Exercises 1 and 2 contain much material which was in the lectures. These exercises are good preprations to solve exercises 3 and 4.

1 *Consider in* $\mathbf{E}^2$ *ellipse*

$$\frac{x^2}{25} + \frac{y^2}{9} = 1\,. \tag{1.1}$$

Find foci of this ellipse

Choose focal polar coordinates for this ellipse and write down the equation of this ellipse in these polar coordinates.

An ellipse is defined analytically in canonical Cartesian coordinates by equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \quad \text{canonical equation}\,. \tag{1.1a}$$

Centre of an ellipse defined by this canonical equation is at the origin. Comparing this equation with equation (1.1) we see that for the ellipse (1.1)

$$a = \text{length of semi-major axis} = 5\,, \quad b = \text{length of semi-minor axis} = 3\,. \tag{1.2}$$

Foci are at the points $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = 4$,

$$F_1 = (-c, 0) = (-4, 0)\,, F_2 = (c, 0) = (4, 0)\,, \quad c = \sqrt{a^2 - b^2}\,. \tag{1.3}$$

Another way to calculate foci. Major axis of the ellipse is equal to $a = 5$ it is on the X axis, hence foci are on the X axis also. Centre of this ellipse is at the origin, hence foci are at the points $F_{2,1} = (\pm c, 0)$. By the geometrical definition of an ellipse we know that the sum of distances from arbitrary point of ellipse to foci is equal to constant. (The ellipse is locus C: $\{K\colon |KF_1| + |KF_2| = \mathit{constant}\}$.) Take two points at the ellipse (1.1), the point $A = (5, 0)$, the vertex on the X axis, and the point $B = (0, 3)$ the upper vertex at the Y axis we have:

$$|A-F_1|+|A-F_2| = |a-c|+|a-(-c)| = 2a = 10 = |B-F_1|+|B-F_2| = 2\sqrt{c^2 + b^2}\,. \tag{1.4}$$

This implies that $c = 4$, thus $F_1 = (-4, 0)$ and $F_2 = (4, 0)$.

To choose focal polar coordinates we take the origin of polar coordinates the left focus, the point $F_1 = (-4, 0)$ and the angle φ is the angle between axis OX and the radius-vector $\mathbf{r}$: $\begin{cases} x = -4 + r\cos\varphi \\ y = r\sin\varphi \end{cases}$. We have $|KF_1| = r$ and according to cosine formula for triangle

$$|KF_2| = \sqrt{r^2 - 2r|F_1F_2| + |F_1F_2|^2}\,.$$

The condition that for arbitrary point K of the ellipse $|KF_1|+|KF_2| = 2a \equiv 10$, becomes:

$$|KF_1| + |KF+2| = r + \sqrt{r^2 - 2r|F_1F_2| + |F_1F_2|^2} = r + \sqrt{r^2 - 2\cdot 8\cdot r\cos\varphi + 8^2} = 10 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{r^2 - 16r\cos\varphi + 64} = 10 - r \Leftrightarrow \tag{1.5}$$

$$r^2 - 16\cos\varphi + 64 = 100 - 20r + r^2 \Leftrightarrow r = \frac{36}{20 - 16\cos\varphi} = \frac{9/5}{1 - 4/5\cos\varphi}\,.$$

We see that in polar coordinates the equation of the ellipse (1.1) is

$$r = \frac{p}{1 - e\cos\varphi}\,, \quad \text{with } p = \frac{a^2 - c^2}{a} = \frac{b^2}{a} = \frac{9}{5} \text{ and } e = \frac{c}{a} = \frac{4}{5}\,. \tag{1.6}$$

Remark Note that taking square in equation (1.5) we come to equivalent equation because $10 - r$ is non-negative: r is less or equal to the sum of distances from a point to foci, and this sum is equal to 10.

2 *Consider a curve in* $\mathbf{E}^2$ *defined in polar coordinates* (r, φ) *by the equation*

$$r = \frac{p}{1 - e\cos\varphi}\,, \quad p > 0\,. \tag{2.1}$$

a) Write down the equation of this curve in Cartesian coordinates $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ *in the case if* $p = 2, e = \frac{1}{3}$*, show that this curve is an ellipse, and find the foci and the centre of this ellipse. Calculate the area of this ellipse.*

b) How does the curve defined by equation (2.1) look in the case if $e = 1$*?*

a) Consider arbitrary parameter p and e obeying the conditions

$$p > 0\,, \quad 0 \le e < 1 \tag{2.2}$$

We have

$$r - re\cos\varphi = p \Leftrightarrow \sqrt{x^2 + y^2} = p + ex \tag{2.3}$$

Take a square of left and right and sides of this equation:

$$x^2+y^2 = p^2+2pex+e^2x^2 \Leftrightarrow (1-e^2)x^2-2pex+y^2 = p^2 \Leftrightarrow (1-e^2)\left(x - \frac{2pe}{1-e^2} + \left(\frac{pe}{1-e^2}\right)^2\right)+y^2 =$$

$$= p^2 + \frac{p^2e^2}{1-e^2} \Leftrightarrow (1-e^2)\left(x - \frac{pe}{1-e^2}\right)^2 + y^2 = p^2 + \frac{p^2e^2}{1-e^2} = \frac{p^2}{1-e^2}\,. \tag{2.4}$$

Hence translating Cartesian coordinates x: $x' = x - \frac{pe}{1-e^2}$ we come to Cartesian coordinates, such that the curve has canonical equation (see equation (1,1a)) in these coordinates[1]:

$$(1-e^2)x'^2 + y^2 = \frac{p^2}{1-e^2} \Leftrightarrow \left(\frac{x'}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\,, \text{with } a = \frac{p}{1-e^2}\,, b = \frac{p}{\sqrt{1-e^2}} \tag{2.5}$$

We come in equation (2.5) to analytical definition of ellipse. The foci of this ellipse in canoncial coordinates x', y' are at the points $x' = \pm c, y' = 0$ (see equation (1.3)) hence x, y coordinates of the foci are

$$y = 0\,, x = \pm c + \frac{p}{1-e^2} e = \pm c + a \cdot \left(\frac{c}{a}\right) = c \pm c = 2c \text{ or } 0\,. \tag{2.5a}$$

We see that in coordinates (x, y) one of the foci is at the origin and the second focus is at the point $(2c.0)$.

Centre of this ellipse is at the origin in Cartesian coordinates (x', y), hence in coordinates (x, y) it is at the middle point between foci, at the point $(c, 0)$.

Thus we see that equation (2.1) describes the ellipse with foci at origi and at the point $(0, 2c)$ and with semi-major and semi-minor axis a, b defined by equation (2.5).

(The parameters, a and c were calculated in equation above).

Now calculate the area of the ellipse. The area of the ellipse is equal to πab. According equations above we can express these parameters in terms of parameters p, e:

$$S = \pi ab = \pi \cdot \frac{p}{1-e^2} \cdot \frac{p}{\sqrt{1-e^2}} = \frac{\pi p^2}{(1-e^2)^{\frac{3}{2}}}\,. \tag{2.6}$$

b): In the case if $e = 1$ we have

$$C: \quad r = \frac{p}{1-\cos\varphi}\,, p > 0\,.$$

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[1] **Remark** You may be suspicious about equivalence of equation (2.3) and equation (2.4). We come to equation (2.4) taking square of equation (2.3). One can easy to see that indeed these equations are equivalent since $0 \le e < 1$. Indeed equation (2.5) (which is evidently equivalent to equation (2.4)) implies that $|x'| = \left|x - \frac{pe}{1-e^2}\right| \le a = \frac{p}{1-e^2}$. This implies that

$$p + ex > p + \left(\frac{pe}{1-e^2} - \frac{p}{1-e^2}\right) e = \frac{p}{1+e} > 0\,. \tag{2.3a}$$

One can see it also immediately after taking square of equation (2.3):we have $y^2 = (p + ex)^2 - x^2 = (p + (e+1)x)(p - (1-e)x)$. Thus $(p + (e+1)(p - (1-e)x) > 0$. Bearing in mind that $0 < e < 1$ we come to equation (2.3a).

Show that this is parabola. We have $r - r\cos\varphi = p$, i.e. $r = p + r\cos\varphi$, i.e. $\sqrt{x^2+y^2} = p + x$, $(p > 0)$. We have

$$\sqrt{x^2+y^2} = (p+x)^2 \Leftrightarrow x^2+y^2 = (p+x)^2 \Leftrightarrow y^2 = (p+2x)p\,.$$

(These equations are equivalent since equation $x^2+y^2 = (p+x)^2$ is equivalent to equation $y^2 = (p+2x)p$, hence $p+2x > 0$ and $p+x > 0$.) We come to

$$y^2 = (p+2x)x = 2px - p^2\,.$$

This is parabola. In Cartesian coordinates $x' = x - \frac{p}{2}$ it has canonical expression

$$y^2 = 2p\left(x - \frac{p}{2}\right) = 2px'\,.$$

Another way to see it is the following: Consider the line $x = -p$, then we see that C is the locus of the points which is on the same distance from the origin and the line $x = -p$:

$$p + r\cos\varphi = p + \frac{p\cos\varphi}{1-\cos\varphi} = \frac{p}{1-\cos\varphi} = r\,.$$

We see that this is the parabola with focus at the origin, and the directrix $x = -p$.

3 *Let C be a curve, intersection of the plane $\alpha\colon 5z - x = 5$ with the conic surface $M\colon x^2+y^2 = z^2$.*

Let C_{proj} be an orthogonal projection of this curve on the plane $z = 0$.

Show that the curve C_{proj} is an ellipse.

Explain why the curve C is also the ellipse.

Find foci of the curve C. In particular show that the origine, vertex of the conic surface M is a focus of the ellipse C.

Find the areas of the ellipses C and C_{pr}.

We have that

$$C\colon \quad \begin{cases} x^2+y^2 = z^2 \\ z = 1 + \frac{x}{5} \end{cases} \Leftrightarrow C\colon \quad \begin{cases} x^2+y^2 = \left(1+\frac{x}{5}\right)^2 \\ z = 1 + \frac{x}{5} \end{cases}$$

and for orthogonal projection

$$C_{\text{proj}}\colon \quad \begin{cases} x^2+y^2 = \left(1+\frac{x}{5}\right)^2 \\ z = 0 \end{cases} \tag{3.1}$$

The projected curve C_{proj} belongs to the plane $z = 0$ and it is described in this plane by the equation $x^2+y^2 = \left(1+\frac{x}{5}\right)^2$. Transforming we come to

$$x^2+y^2 = \left(1+\frac{x}{5}\right)^2 \Leftrightarrow \frac{24}{25}x^2 - \frac{2x}{5} + y^2 = 1 \Leftrightarrow \frac{24}{25}\left(x^2 - \frac{5}{12}x\right) + y^2 = 1 \Leftrightarrow$$

$$\frac{24}{25}\left(x^2-\frac{5}{12}x+\frac{25}{24^2}\right)+y^2=1+\frac{1}{24}\Leftrightarrow\frac{24}{25}\left(x-\frac{5}{24}\right)^2+y^2=\frac{25}{24}\Leftrightarrow$$

$$\Leftrightarrow\frac{24^2}{25^2}\left(x-\frac{5}{24}\right)^2+\frac{24}{25}y^2=1\,. \tag{3.1a}$$

We see that the curve C_{proj} is an ellipse: it follows from equation (3.1a) that in Cartesian coordinates $\begin{cases} x'=x-\frac{5}{24} \\ y'=y \end{cases}$ equation (3.1) has the appearance

$$\left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2=1\,,\ \text{for}\ a=\frac{25}{24},b=\frac{5\sqrt{6}}{12}\,. \tag{3.1b}$$

The semi-major axis of this ellpse is $a=\frac{25}{24}$, and the semi-minor axis is $b=\sqrt{\frac{25}{24}}=\frac{5\sqrt{6}}{12}$.

According to general formulae (see also the equations (1.1), (1.2) and (1.3) of solution of exercise 1) the foci of this ellipse in coordinates x',y' are at the points

$$y'=0\,,\quad ,x'=\pm c=\pm\sqrt{a^2-b^2}=\sqrt{\left(\frac{25}{24}\right)^2-\frac{25}{24}}=\pm\frac{5}{24}\,.$$

Thus we see that in the initial coordinates $\begin{cases} x=x'+\frac{5}{24} \\ y=y' \end{cases}$ the foci are at the points

$$y=0\,,\quad x=\pm\frac{5}{24}+\frac{5}{24}\,,\text{i.e.}x_1=0,x_2=\frac{10}{24}=\frac{5}{12}$$

We see that according to the Theorem one of the foci is at the origin.

Remark Of course the fact that one of the foci is at the origin follows from the Theorem. One can just use this fact, and find only the second focus of the ellipse using the information about the centre of the ellipse from the equations (3.1a) or (3.1b).

Now return to the curve C.

The curve C is also an ellipse since its orthogonal projection on the plane $z=0$ is the ellipse.

The plane $5z=5+x$ intersects the plane $z=0$ under the angle θ: $\tan\theta=\frac{1}{5}$, i.e. $\cos\theta=\frac{5}{\sqrt{26}}=\frac{5\sqrt{26}}{26}$. Hence the Cartesian coordinates on the plane of this ellipse are

$$\tilde{x}=\frac{x}{\cos\theta}=\frac{\sqrt{26}}{5}x\,,\quad \tilde{y}=y,.$$

The minor axis of the ellipse C has the same length $l=2b$ as the minor axis of the projected ellipse C_{proj}; the major axis of the ellipse C has the length $\frac{2a}{\cos\theta}$, where a is the length of the semi-major axis of the projected ellipse C_{proj}. Recall that $a = \frac{25}{24}$ and $b = \frac{5\sqrt{6}}{12}$ (see equation (3.1b)). Thus we see that the area S of the projected ellipse C_{proj} is equal to

$$S(C_{\text{proj}}) = \pi ab = \pi \cdot \frac{25}{24} \cdot \frac{5\sqrt{6}}{12} = \frac{125\sqrt{6}}{288}\pi\,,$$

and the area S of the ellipse C is equal to

$$S(C) = \pi \frac{ab}{\cos\theta} = \frac{S(C_{\text{proj}})}{\cos\theta} = \frac{125\sqrt{6}}{288}\pi \cdot \frac{\sqrt{26}}{5} = \frac{25\sqrt{39}}{144}\pi\,.$$

4 *Let C be a curve, intersection of the plane $\alpha\colon z = 1 + kx$ (k is real parameter) with the conic surface M: $2x^2 + 2y^2 = 9z^2$.*

Let C_{proj} be an orthogonal projection of this curve on the plane $z = 0$.

Find the values of k such that the curve C and the curve C_{pr} are ellipses.

Find the values of k such that the curve C and curve C_{pr} are parabolas.

In the case if a curve C (and a curve C_{proj}) are parabolas,show that the vertex of the conic surface M, the origin, is the focus of this parabola C_{proj}

Find the directrix of this parabola.

The equations of the curve C and of the curve C_{proj} which is its orthogonal projection on the plane $z = 0$ are

$$C\colon \quad \begin{cases} 2x^2 + 2y^2 = 9z^2 \\ z = 1 + kx \end{cases}, \quad C_{\text{proj}}\colon \quad \begin{cases} 2x^2 + 2y^2 = 9(1 + kx)^2 \\ z = 0 \end{cases} \tag{4.1}$$

It is enough to analyze the curve C_{proj}. We know that the curve C is ellipse, parabola or hyperbola if the curve C_{proj} is ellipse, parabola or hyperbola, respectively.

The projected curve C_{proj} belongs to the plane $z = 0$ and it is described in this plane by the equation $2x^2 + 2y^2 = 9\,(1 + kx)^2$.

Transforming we come to

$$2x^2 + 2y^2 = 9\,(1 + kx)^2 \Leftrightarrow (2 - 9k^2)x^2 - 18kx + 2y^2 = 9\,.$$

If $k = \pm\frac{\sqrt{2}}{3}$ then we see that C_{proj} is parabola

$$C_{\text{proj}}\colon \quad y^2 = \frac{9}{2} + 9kx = \frac{9}{2} \pm 3\sqrt{2}x \quad (z = 0)\,. \tag{4.2}$$

In the case if $k \neq \pm\frac{\sqrt{2}}{3}$ we continue the transformation (2a) of the curve C_{proj}

$$C_{\text{proj}}\colon\ (2-9k^2)x^2-18kx+2y^2 = 9 \leftrightarrow (2-9k^2)\left(x - \frac{9k}{2-9k^2}\right)^2 + 2y^2 = 9 + \frac{81k^2}{2-9k^2}\,. \tag{4.3}$$

We see that if $2-9k^2>0$ then the curve $C_{\rm proj}$ (and respectively the curve C) are ellipses, and if $2-9k^2<0$ then the curve $C_{\rm proj}$ (and respectively the curve C) are hyperbolas. (If $2-9k^2=0$ it is parabola.)

Consider in detail the case of ellipse, $(2-9k^2>0)$.

Notice that in (equation (4.3) the right hand side is positive: $9+\frac{81k^2}{2-9k^2}>9$ if $2-9k^2>0$. Denote it by H^2, $H^2=9+\frac{81k^2}{2-9k^2}$. We can rewrite equation (4.3) in the following way:

$$(2-9k^2)\left(x-\frac{9k}{2-9k^2}\right)^2+2y^2=H^2\,.$$

We see that in Cartesian coordinates $\begin{cases} x'=x-\frac{9k}{2-9k^2} \\ y'=y \end{cases}$ the curve $C_{\rm proj}$ has appearance

$$\left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2=1\,,\ \text{where}\ a^2=\frac{H^2}{2-9k^2}\,,\, b^2=\frac{H^2}{2}\,,\, H^2=9+\frac{81k^2}{2-9k^2}\,,\, (z=0)\,.$$

We see that if $k\colon 2-9k^2>0$ then the curve $C_{\rm proj}$ and the curve C are ellipses.

The semi-major axis of the ellipse $C_{\rm proj}$ has length $a=\frac{H}{\sqrt{2-9k^2}}$, the semi-minor axis of the ellipse $C_{\rm proj}$ has length $b=\frac{H}{\sqrt{2}}=\frac{H\sqrt{2}}{2}$.

Now return to the case of parabola (see equation (4.2)):

In fact we have two parabolas, but trasformation $x\mapsto -x$ transform them to each other. We consider only one of these parabola: for $k=\frac{\sqrt{2}}{3}$:

$$C_{\rm proj}\colon\quad y^2=\frac{9}{2}+3\sqrt{2}x\ (z=0)\,. \tag{4.4}$$

How to find the focus of this parabola? Do it in two ways:

`I-st way` We know that the focus of the parabola $y^2=\frac{9}{2}+3\sqrt{2}x$ is the vertex of conical surface: $F=(0,0)$.

Find the vertex of the parabola: if $y=0$ then $x=-\frac{3\sqrt{2}}{4}$. Hence the vertex of the parabola is at the point $\left(-\frac{3\sqrt{2}}{4},0\right)$. The vetex is at the middle point between the intersection of directrix with OX axis and the focus. Hence the directrix is $x=-\frac{3\sqrt{2}}{2}$.

`II-nd way` Choose canonical Cartesian coordinates for this parabola. In canonical Cartesian coordinates this parabola looks like $y'^2=2px'$. Equation (4.4) for this parabola states that

$$y^2=3\sqrt{2}\left(x+\frac{3\sqrt{2}}{4}\right)\,.$$

We see that in the new Cartesian coordinates $\begin{cases} x'=x+\frac{3\sqrt{3}}{4} \\ y'=y \end{cases}$ this parabola looks canonically:

$$y'^2=2px'\,,\quad p=\frac{3\sqrt{2}}{2}$$

Hence in the canonical Cartesian coordinates (x', y') the focus of the parabola is at the point $x' = \frac{p}{2} = \frac{3\sqrt{2}}{4}$, $y' = 0$ and the directrix is $x' = -\frac{p}{2} = -\frac{3\sqrt{2}}{4}$. Hence in initial Cartesian coordinates we have for the focus of parabola

$$F = (x_F, y_F) = (x'_F - \frac{3\sqrt{2}}{4}, 0) = (0, 0)\,.$$

This is in the accordance with the statement that the vertex of the cone is the focus of the parabola; and for directrix we have

$$x = x' - \frac{3\sqrt{2}}{4} = -\frac{3\sqrt{2}}{2}\,.$$

5 *Find foci and directrix of the parabola $y = ax^2$, $(a > 0)$.*

One way to do this exercise is to find canonical coordinates. We can find focus of parabola straightforwardly. Focus of parabola $y = ax^2$ belongs to OY axis, $F = (0, f)$ and directrix is $y = -c$. Since for every point of parabola the distances coincide we have:

$$\sqrt{x^2 + (y - f)^2} = y + c\,.$$

Since $y = ax^2 > 0$ this equation is equivalent to the equation:

$$x^2 + y^2 - 2fy + f^2 = y^2 + 2yc + c^2\,, \quad \text{for } y = ax^2\,,$$

i.e.

$$x^2 = 2y(c + f) + c^2 - f^2 = 2ax^2(c + f) + c^2 - f^2$$

This relation holds for all x. Hence $c = f$ (directrix is $y = -c$) and $4af = 1$. We come to answer $F = \left(0, \frac{1}{4a}\right)$.