Riemannian Geometry

it is a draft of Lecture Notes of H.M. Khudaverdian.
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1 Riemannian manifolds

1.1 Manifolds. Tensors. (Recollection)

1.1.1 Manifolds

I recall briefly basics of manifolds and tensor fields on manifolds.

An n-dimensional manifold $M = M^n$ is a space[1]

such that in a vicinity of an arbitrary point one can consider local coordinates $\{x^1, \dots, x^n\}$. (We say that in a vicinity of this point a manifold M is covered by local coordinates $\{x^1, \dots, x^n\}$). One can consider different local coordinates. If coordinates $\{x^1, \dots, x^n\}$ and $\{x^{1'}, \dots, x^{n'}\}$ both are defined in a vicinity of the given point then they are related by *bijective transition functions* which are defined on domains in $\mathbf{R}^n$ and taking values also in $\mathbf{R}^n$:

$$\begin{cases} x^{1'} = x^{1'}(x^1, \dots, x^n) \\ x^{2'} = x^{2'}(x^1, \dots, x^n) \\ \dots \\ x^{n-1'} = x^{n-1'}(x^1, \dots, x^n) \\ x^{n'} = x^{n'}(x^1, \dots, x^n) \end{cases} \tag{1.1}$$

We say that n-dimensional manifold is *differentiable* or *smooth* if all transition functions are diffeomorphisms, i.e. they are smooth. Invertability implies that Jacobian matrix is non-degenerate:

$$\det \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \dots & \frac{\partial x^{2'}}{\partial x^n} \\ \dots & & \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \neq 0\,. \tag{1.2}$$

(If bijective function $x^{i'} = x^{i'}(x^i)$ is smooth function, and its inverse, the transition function $x^i = x^i(x^i)$ is also smooth function, then matrices $||\frac{\partial x^{i'}}{\partial x^i}||$ and $||\frac{\partial x^i}{\partial x^{i'}}||$ are both well defined, hence condition (1.2) is obeyed.

[1] A space M is a topological space, i.e. it is covered by a collection $\mathcal{F}$ of sets, which are called *open* sets. This collection obeys the following axioms

i) the union of an arbitrary set of open sets is an open set

ii) the intersection of finite number of open sets is an open set

iii) the whole space M and the empty set $\emptyset$ are open sets

Example

open domain in $\mathbf{E}^n$

A good example of manifold is an open domain D in n-dimensional vector space $\mathbf{R}^n$. Cartesian coordinates on $\mathbf{R}^n$ define global coordinates on D. On the other hand one can consider an arbitrary local coordinates in different domains in $\mathbf{R}^n$. E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain $D = \{x, y\colon\, y > 0\}$ of $\mathbf{R}^2$ defined by standard formulae:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \tag{1.3}$$

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r \tag{1.4}$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z\colon\, x > 0, y > 0, z > 0\}$ of $\mathbf{R}^3$ (or in other domain of $\mathbf{R}^3$) defined by standard formulae

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases},$$

,

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2\sin\theta \tag{1.5}$$

Choosing domain where polar (spherical) coordinates are well-defined we have to be aware that coordinates have to be well-defined and transition functions (1.1) have to obey condition (1.2), i.e. they have to be diffeomorphisms. E.g. for domain D in example (1.3) Jacobian (1.4) does not vanish if and only if $r > 0$ in D.

Consider another examples of manifolds, and local coordinates on manifolds.

Example

Circle S^1 in $\mathbf{E}^2$

Consider circle $x^2 + y^2 = R^2$ of radius R in $\mathbf{E}^2$.

One can consider on the circle different local coordinates

i) *polar coordinate* φ:

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}, \qquad 0 < \varphi < 2\pi$$

(this coordinate is defined on all the circle except a point $(R, 0)$),

ii) *another polar coordinate* φ':

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}, \qquad -\pi < \varphi < \pi\,,$$

this coordinate is defined on all the circle except a point $(-R, 0)$,

iii) stereographic coordinate t with respect to north pole of the circle

$$\begin{cases} x = \frac{2R^2 t}{t^2+R^2} \\ y = R\frac{t^2-R^2}{t^2+R^2} \end{cases}, \quad t = \frac{Rx}{R-y}\,, \tag{1.6}$$

this coordinate is defined at all the circle except the north pole,

iiii) stereographic coordinate t' with respect to south pole of the circle

$$\begin{cases} x = \frac{2R^2 t'}{t'^2+R^2} \\ z = R\frac{R^2-t'^2}{t'^2+R^2} \end{cases}, \quad t' = \frac{Rx}{R+y}\,,$$

this coordinate is defined at all the points except the south pole.

We considered four different local coordinates on the circle S^1. Write down some transition functions (1.1) between these coordinates

- polar coordinate φ coincide with polar coordinate φ' in the domain $x^2 + y^2 > 0$, and in the domain $x^2 + y^2 < 0$ $\varphi' = \varphi - 2\pi$.
- Transition function from polar coordinate φ to stereographic coordinates t is $t = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$,
- transition function from stereographic coordinate t to stereographic coordinate t' is

$$t' = \frac{R^2}{t}\,,$$

(see Homework 0.)

Example

Sphere S^2 in $\mathbf{E}^3$

Consider sphere $x^2+y^2+z^2=R^2$ of radius a in $\mathbf{E}^3$.
One can consider on the sphere different local coordinates
i) *spherical coordinates on domain of sphere θ,φ*:

$$\begin{cases} x=a\sin\theta\cos\varphi \\ y=a\sin\theta\sin\varphi \\ z=a\cos\theta \end{cases} \quad , \quad 0<\theta<\pi\,, -\pi<\varphi<\pi$$

ii) stereographic coordinates u,v with respect to north pole of the sphere

$$\begin{cases} x=\frac{2a^2u}{a^2+u^2+v^2} \\ y=\frac{2a^2v}{a^2+u^2+v^2} \\ z=a\frac{u^2+v^2-a^2}{a^2+u^2+v^2} \end{cases} \quad , \quad \frac{x}{u}=\frac{y}{v}=\frac{a-z}{a}\,, \quad \begin{cases} u=\frac{ax}{a-z} \\ v=\frac{ay}{a-z} \end{cases} \quad .$$

iii) stereographic coordinates u',v' with respect to south pole of the sphere

$$\begin{cases} x=\frac{2a^2u'}{a^2+u'^2+v'^2} \\ y=\frac{2a^2v}{a^2+u'^2+v'^2} \\ z=a\frac{a^2-u'^2-v'^2}{a^2+u^2+v^2} \end{cases} \quad , \quad \frac{x}{u'}=\frac{y}{v'}=\frac{a+z}{a}\,, \quad \begin{cases} u'=\frac{ax}{a+z} \\ v'=\frac{ay}{a+z} \end{cases} \quad .$$

(see also Homework 0)

Spherical coordinates are defined elsewhere except poles and the meridians $y=0, x\leq 0$.

Stereographical coordinates (u,v) are defined elsewhere except north pole;

stereographic coordinates (u',v') are defined elsewhere except south pole.

One can consider transition function between these different coordinates.E.g. transition functions from spherical coordinates i) to stereographic coordinates (u,v) are

$$\begin{cases} u=\frac{ax}{a-z}=\frac{a\sin\theta\cos\varphi}{1-\cos\theta}=a\mathrm{cotan}\,\frac{\theta}{2}\cos\varphi \\ v=\frac{ay}{a-z}=\frac{a\sin\theta\sin\varphi}{1-\cos\theta}=a\mathrm{cotan}\,\frac{\theta}{2}\sin\varphi \end{cases} \quad ,$$

and transition function from stereographic coordinates u,v to stereographic coordinates (u',v') are

$$\begin{cases} u'=\frac{a^2u}{u^2+v^2} \\ v'=\frac{a^2v}{u^2+v^2} \end{cases} \quad ,$$

(see Homework 0.)

Remark

† One very important property of stereographic projetion which we do not use in this course but it is too beautiful not to mention it: under stereographic projection all points of the circle of radius $R = 1$ with rational coordinates x and y and only these points transform to rational points on line. Thus we come to Pythagorean triples $a^2 + b^2 = c^2$. The same is for unit sphere: the stereographic projection establishes one-one correspondence between points on the unit sphere with rational coordinates and rational points on the plane.

1.1.2 Tensors on Manifold

tangent vector and tangent vector space

Tangent vector at the given point can be considered as a derivation of function at this point. For an arbitrary (smooth) function f defined in a vicinity of a given point $\mathbf{p}$ a tangent vector $\mathbf{A}(x) = A^i(x)\frac{\partial}{\partial x^i}$ defines the directional derivative of this function

$$\mathbf{A}\colon \quad f \mapsto \partial_{\mathbf{A}} f\big|_{\mathbf{p}} = A^i(x)\frac{\partial f}{\partial x^i}\big|_{\mathbf{p}} .$$

Using the chain rule one can see that under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i} = A^i(x)\frac{\partial x^{i'}(x)}{\partial x^i}\frac{\partial}{\partial x^{i'}} = A^{i'}(x'(x))\frac{\partial}{\partial x^{i'}} ,$$

i.e.

$$A^{i'}(x') = \frac{\partial x^{i'}}{\partial x^i}A^i(x) . \tag{1.7}$$

This leads as to the following equivalent definition of the tangent vector.

Definition Let $M = M^n$ be n-dimensional manifold, and $\mathbf{p}$ the point on it. To define a vector $\mathbf{A}$ tangent to the manifold at the point $\mathbf{p}$ we assign to an arbitrary given local coordinates $\{x^i\}$ the array $\{A^i\}$ $(i = 1, \dots, n)$ of numbers (components) such that under changing of local coordinates this array transforms according to equation (1.7):

$$\begin{array}{ccc} \texttt{coordinates} & & \texttt{components of vector} \\ \{x^i\} & \rightarrow & \{A^i\} \\ \{x^{i'}\} & \rightarrow & \{A^{i'}\} \end{array} \qquad \text{such that } A^{i'} = \frac{\partial x^{i'}(x)}{\partial x^i}\big|_{\mathbf{p}}A^i . \tag{1.8}$$

Tangent vector space $T_{\mathbf{p}}M$ at the point $\mathbf{p}$ is the space of vectors tangent to the manifold at the point M.

1 *-form (covector) in a given point*

We defined above vectors of tangent space $T_{\mathbf{p}}M$. Now we consider dual obejcts: we consider cotangent space $T^*_{\mathbf{p}}M$ (for every point $\mathbf{p}$ on manifold M)— space of linear functions on tangent vectors, i.e. space of 1-forms which sometimes are called *covectors.*:

Linear function, 1-form $\omega = \omega_i dx^i$ is a function on tangent vectors:

$$T_{\mathbf{p}}M \ni \mathbf{A} = A^i \frac{\partial}{\partial x^i}\,, \omega\left(\mathbf{A}\right) = \omega_m dx^m \left(A^i \frac{\partial}{\partial x^i}\right) = \omega_m A^i dx^i \underbrace{\left(\frac{\partial}{\partial x^m}\right)}_{\delta_i^m} = \omega_m A^m\,.$$

If we consider new coordinates $x^{i'} = x^{i'}(x)$, then

$$\omega = \omega_i dx^i = \omega_i \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'}\right) = \underbrace{\omega_i \frac{\partial x^i}{\partial x^{i'}}}_{\omega_{i'}} dx^{i'}$$

i.e., nne-form (covector) $\omega = \omega_i(x) dx^i$ transforms as follows

$$\omega_{m'}(x') = \frac{\partial x^m(x')}{\partial x^{m'}} \omega_m(x)\,. \tag{1.9}$$

Differential form sometimes is called *covector*.

In the same way as for vectors we may give definition of covectors in the following way:

Definition Let $M = M^n$ be n-dimensional manifold, and $\mathbf{p}$ the point on it. To define a *covector* $\mathbf{A}$ at the point $\mathbf{p}$, (the linear function on tangent vectors at $\mathbf{p}$) we assign to an arbitrary given local coordinates $\{x^i\}$ the collection $\{\omega_i\}$ $(i = 1, \ldots, n)$ of numbers (components) such that under changing of local coordinates this collection transforms according to equation (1.9):

$$\begin{array}{ccc} \texttt{coordinates} & & \texttt{components of covector} \\ \{x^i\} & \rightarrow & \{\omega_i\} \\ & & \\ \{x^{i'}\} & \rightarrow & \{\omega_{i'}\} \end{array} \quad \text{such that}\, \omega_{i'} = \frac{\partial x^i(x')}{\partial x^{i'}}\Big|_{\mathbf{p}} \omega_i\,. \tag{1.10}$$

Remark Notice the difference between formulae (1.7) and (1.9). In formulae (1.7), (1.8) transformation is performed by matrix of derivatives

$\partial x^{i'}\partial x^i$ from coordinates x^i to the new coordinates $x^{i'}$, and in formula (1.9) transformation is performed by the *inverse* matrix, matrix of derivatives $\partial x^i \partial x^{i'}$ from new coordinates $x^{i'}$ to the initial coordinates x^i.

Tensors:

Definition Consider geometrical object such that in arbitrary local coordinates (x^i) it is given by components

$$Q = \left\{ Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \frac{\partial}{\partial x^{i_1}}(x) \right\}, i_1, \dots, i_p; j_1, \dots, j_q = 1, 2, \dots, n,$$

and under changing of coordinates this object is transformed in the following way:

$$Q^{i'_1 i'_2 \dots i'_p}_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x). \tag{1.11}$$

We say that this is *p-times contravariant, q-times covariant tensor of valence* $\begin{pmatrix} p \\ q \end{pmatrix}$, *or shorter, tensors of the type* $\begin{pmatrix} p \\ q \end{pmatrix}$.

`Caution: this tensor possess` n^{p+q} `components.`

Sometimes it is useful to view $\begin{pmatrix} p \\ q \end{pmatrix}$-tensor as

$$Q = Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} dx^{j_1} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_q}$$

(Compare with definition of vector: $\mathbf{A} = A^i \frac{\partial}{\partial x^i}$ and covector (1-form) $\omega = \omega_i dx^i$).

Examples

Note that vector field (1.7) is nothing but tensor field of valency $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and 1-form (1.9) is nothing but tensor field of valency $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

One can consider *contravariant* tensors of the rank p

$$T = T^{i_1 i_2 \dots i_p}(x) \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}}$$

with components $\{T^{i_1 i_2 \dots i_k}\}(x)$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ (see (1.1)) they transform as follows:

$$T^{i'_1 i'_2 \dots i'_p}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} T^{i_1 i_2 \dots i_p}(x). \tag{1.12}$$

One can consider *covariant* tensors of the rank q

$$S = S_{j_1 j_2 \dots j_q} dx^{j_1} \otimes dx^{j_2} \otimes \dots dx^{j_q}$$

with components $\{S_{j_1 j_2 \dots j_q}\}$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ they transform as follows:

$$S_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \frac{\partial x^{i_2}}{\partial x^{i'_2}} \dots \frac{\partial x^{i_p}}{\partial x^{i'_p}} S_{j_1 j_2 \dots j_q}(x) \,.$$

E.g. if S_{ik} is a covariant tensor of rank 2 then

$$S_{i'k'}(x') = \frac{\partial x^i(x')}{\partial x^{i'}} \frac{\partial x^k(x')}{\partial x^{k'}} S_{ik}(x) \,. \tag{1.13}$$

If A^i_k is a tensor of rank $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (linear operator on $T_{\mathbf{p}}M$) then

$$A^{i'}_{k'}(x') = \frac{\partial x^{i'}(x)}{\partial x^i} \frac{\partial x^k(x')}{\partial x^{k'}} A^i_k(x) \,.$$

Remark *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention.* it implies that when an index occurs twice in the same expression in upper and in lower postitions, then the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

Using Einstein summation rules we avoid to write bulky expressions. Later we will see that these notations are really very effective. E.g. equation (1.7) in 'standard' notations will appear as

$$\text{for every } i' = 1, \dots, n \;\; A^{i'}(x') = \sum_{i=1}^{n} \frac{\partial x^{i'}}{\partial x^i} A^i(x) \,.$$

1.2 Riemannian manifold

1.2.1 Riemannian manifold— manifold equipped with Riemannian metric

Definition The Riemannian manifold (M, G) is a manifold equipped with a Riemannian metric.

The Riemannian metric G on the manifold M defines the length of the tangent vectors and the length of the curves.

Definition Riemannian metric G on n-dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$.

It means that in every coordinate system $(x^1, \dots, x^n)$ a metric $G = g_{ik}dx^i dx^k$ is defined by a matrix valued smooth function $g_{ik}(x)$ $(i = 1, \dots, n; k = 1, \dots n)$ such that for any two vectors

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i},\ \mathbf{B} = B^i(x)\frac{\partial}{\partial x^i},$$

tangent to the manifold M at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \dots, x^n)$ $(\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M)$ the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B} \rangle_G \big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x) g_{ik}(x) B^k(x) =$$

$$\begin{pmatrix} A^1 \dots A^n \end{pmatrix} \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \dots & \dots & \dots \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \tag{1.14}$$

where

- $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
- $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.
  $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \dots = u^n = 0$ (positive-definiteness)
- $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$, i.e. $g_{ik}(x)$ are smooth functions.

The matrix $||g_{ik}||$ of components of the metric G we also sometimes denote by G.

Now we establish rule of transformation for entries of matrix $g_{ik}(x)$, of metric G.

Notice that an arbitrary matrix entry g_{ik} is nothing but scalar product of vectors ∂_i, ∂_k at the given point:

$$g_{ik}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle, \quad \text{in coordinates } (x^1, \dots, x^n) \tag{1.15}$$

Use this formula for establishing rule of transformations of $g_{ik}(x)$. In the new coordinates $x^{i'} = (x^{1'}, \dots, x^{n'})$ according this formula we have that

$$g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle, \quad \text{in coordinates } (x^1, \dots, x^n)\,.$$

Now using chain rule, linearity of scalar product and formula (1.15) we see that

$$g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle = \left\langle \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i}, \frac{\partial x^k}{\partial x^{k'}} \frac{\partial}{\partial x^k} \right\rangle$$
$$= \frac{\partial x^i}{\partial x^{i'}} \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle}_{g_{ik}(x)} \frac{\partial x^k}{\partial x^{k'}} = \frac{\partial x^i}{\partial x^{i'}} g_{ik}(x) \frac{\partial x^k}{\partial x^{k'}} \tag{1.16}$$

This transformation law implies that g_{ik} entries of matrix $||g_{ik}||$ are components of *covariant tensor field* $G = g_{ik} dx^i dx^k$ *of rank 2*(see equation (1.13)).

One can say that Riemannian metric is defined by symmetric covariant smooth tensor field G of the rank 2 which defines scalar product in the tangent spaces $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$. Components of tensor field G in coordinate system are functions $g_{ik}(x)$:

$$G = g_{ik}(x) dx^i \otimes dx^k\,,$$
$$\langle \mathbf{A}, \mathbf{B} \rangle = G(\mathbf{A}, \mathbf{B}) = g_{ik}(x) dx^i \otimes dx^k \,(\mathbf{A}, \mathbf{B})\;. \tag{1.17}$$

In practice it is more convenient to perform transformation of metric G under changing of coordinates in the following way:

$$G = g_{ik} dx^i \otimes dx^k = g_{ik} \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'} \right) \otimes \left(\frac{\partial x^k}{\partial x^{k'}} dx^{k'} \right) =$$
$$\frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} dx^{i'} \otimes dx^{k'} = g_{i'k'} dx^{i'} \otimes dx^{k'}\,, \text{ hence } g_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}}\,. \tag{1.18}$$

We come to transfomation rule (1.16).

Later by some abuse of notations we sometimes omit the sign of tensor product and write a metric just as

$$G = g_{ik}(x) dx^i dx^k\,.$$

1.2.2 Examples

- $\mathbf{R}^n$ with canonical coordinates $\{x^i\}$ and with metric

$$G = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$$

$G = ||g_{ik}|| = \text{diag}\,[1, 1, \ldots, 1]$

Recall that this is a basis example of n-dimensional Euclidean space $\mathbf{E}^n$, where scalar product is defined by the formula:

$$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k = X^1 Y^1 + X^2 Y^2 + \cdots + X^n Y^n \,.$$

In the general case if $G = ||g_{ik}||$ is an arbitrary symmetric positive-definite metric then $G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k$. One can show that there exists a new basis $\{\mathbf{e}_i\}$ such that in this basis $G(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik}$. This basis is called orthonormal basis. (See the Lecture notes in Geometry)

Scalar product in vector space defines the *same* scalar product at all the points. In general case for Riemannian manifold scalar product depends on a point. In Riemannian manifold we consider arbitrary transformations from local coordinates to new local coordinates.

- Euclidean space $\mathbf{E}^2$ with polar coordinates in the domain $y > 0$ ($x = r\cos\varphi, y = r\sin\varphi$):

$dx = \cos\varphi dr - r\sin\varphi d\varphi, dy = \sin\varphi dr + r\cos\varphi d\varphi$. In new coordinates the Riemannian metric $G = dx^2 + dy^2$ will have the following appearance:

$$G = (dx)^2 + (dy)^2 = (\cos\varphi dr - r\sin\varphi d\varphi)^2 + (\sin\varphi dr + r\cos\varphi d\varphi)^2 = dr^2 + r^2(d\varphi)^2$$

We see that for matrix $G = ||g_{ik}||$

$$\underbrace{G = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{in Cartesian coordinates}}, \qquad \underbrace{G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}_{\text{in polar coordinates}}$$

- Circle

Interval $[0, 2\pi)$ in the line $0 \le x < 2\pi$ with Riemannian metric

$$G = a^2 dx^2 \tag{1.19}$$

Renaming $x \mapsto \varphi$ we come to habitual formula for metric for circle of the radius a: $x^2+y^2=a^2$ embedded in the Euclidean space $\mathbf{E}^2$:

$$G = a^2 d\varphi^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \end{cases} , 0 < \varphi < 2\pi, \quad \text{or} \quad -\pi < \varphi < \pi\,. \tag{1.20}$$

Rewrite this metric in stereographic coordinate t:

$$G = a^2 d\varphi^2 = 4a^4 dt^2 (a^2+t^2)^2\,, \quad \text{where } t = \frac{ax}{a-y} = \frac{a^2\cos\varphi}{a-a\sin\varphi} = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\,. \tag{1.21}$$

(See (1.6) and Homeworks 0 and 2.)

- Cylinder surface

  Consider domain in $\mathbf{R}^2$, $D = \{(x,y):\,,\ 0 \le x < 2\pi$ with Riemannian metric

$$G = a^2 dx^2 + dy^2 \tag{1.22}$$

  We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard polar coordinates for cylinder surface of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\varphi^2 + dh^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} , 0 < \varphi < 2\pi, -\infty < h < \infty \tag{1.23}$$

  (Coordinate φ is well defined for $-\pi < \varphi < \pi$ also.)

- Sphere

  Consider domain in $\mathbf{R}^2$, $0 < x < 2\pi$, $0 < y < \pi$ with metric $G = dy^2 + \sin^2 y dx^2$ We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard spherical coordinates for sphere $x^2+y^2+z^2=a^2$ of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2 \qquad \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases} , \quad 0 < \theta < \pi\,,\ 0 < \varphi < 2\pi\,. \tag{1.24}$$

(See examples also in the Homeworks.)

If we omit the condition of positive-definiteness for Riemannian metric we come to so called *Pseudoriemannian metric.* Manifold equipped with pseudoriemannian metric is called pseudoriemannian manifold. Pseudoriemannian manifolds appear in applications in the special and general relativity theory.

In pseudoriemanninan space scalar product $(\mathbf{X}, \mathbf{X})$ may take an arbitrary real values: it can be positive, negative, it can be equal to zero. Vectors $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{X}) = 0$ are called null-vectors.

For example consider 4-dimensional linear space $\mathbf{R}^4$ with pseudometric

$$G = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 .$$

For an arbitrary vector $\mathbf{X} = (a^0, a^1, a^2, a^3)$ scalar product $(\mathbf{X}, \mathbf{X})$ is positive if $(a^0)^2 > (a_1)^2 + (a_2)^2 + (a_3)^2$, and it is negative if $(a^0)^2 < (a_1)^2 + (a_2)^2 + (a_3)^2$, and $\mathbf{X}$ is null-vector if $(a^0)^2 = (a_1)^2 + (a_2)^2 + (a_3)^2$. It is so called Minkovski space. The coordinate x^0 plays a role of the time: $x^0 = ct$, where c is the value of the speed of the light. Vectors $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{X}) > 0$ are called time-like vectors and they called space-like vectors if $(\mathbf{X}, \mathbf{X}) < 0$.

1.2.3 Scalar product $\rightarrow$ Length of tangent vectors and angle between them

The Riemannian metric defines scalar product of tangent vectors attached at the given point. Hence it defines the length of tangent vectors and angle between them. If $\mathbf{X} = X^m \frac{\partial}{\partial x^m}, \mathbf{Y} = Y^m \frac{\partial}{\partial x^m}$ are two tangent vectors at the given point $\mathbf{p}$ of Riemannian manifold with coordinates $x^1, \dots, x^n$, then we have that lengths of these vectors equal to

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x) X^i X^k}, \quad |\mathbf{Y}| = \sqrt{\langle \mathbf{Y}, \mathbf{Y} \rangle} = \sqrt{g_{ik}(x) Y^i Y^k}, \tag{1.25}$$

and an 'angle' θ between these vectors is defined by the relation

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}| \cdot |\mathbf{Y}|} = \frac{g_{ik} X^i Y^k}{\sqrt{g_{ik}(x) X^i X^k} \sqrt{g_{ik}(x) Y^i Y^k}} \tag{1.26}$$

Remark We say 'angle' but we calculate just cosinus of angle.

Example Let M be 3-dimensional Riemannian manifold, and $\mathbf{p} \in M$ a point in it. Suppose that the manifold M is equipped with local coordinates x, y, z in a vicinity of this point, and the expression of Riemannian metric in these local coordinates is

$$G = \frac{dx^2 + dy^2 + dz^2}{(1 + x^2 + y^2 + z^2)^2} . \tag{1.27}$$

Consider the vectors $\mathbf{X} = a\partial_x + b\partial_y + c\partial_z$ and $\mathbf{Y} = p\partial_x + q\partial_y + r\partial_z$, attached at the point $\mathbf{p}$, with coordinates $x = 2, y = 2, z = 1$. Find the lengths of vectors $\mathbf{X}$ and $\mathbf{Y}$ and find cosinus of the angle between these vectors.

We see that matrix of Riemannian metric is

$$||g_{ik(x)}|| = \begin{pmatrix} \frac{1}{(1+x^2+y^2+z^2)^2} & 0 & 0 \\ 0 & \frac{1}{(1+x^2+y^2+z^2)^2} & 0 \\ 0 & 0 & \frac{1}{(1+x^2+y^2+z^2)^2} \end{pmatrix} \text{ i. e. } g_{ik}(x,y,z) = \frac{\delta_{ik}}{(1+x^2+y^2+z^2)^2},$$

where $g_{ik}(x)$ are entries of matrix: $G = g_{ik}(x)dx^i dx^k$, (δ_{ik} is Kronecker symbol: $\delta_{ik} = 1$ if $i = k$ and it vanishes otherwise).

According to formulae above

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x,y,z)X^i X^k}\big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik}X^iX^k}{(1+x^2+y^2+z^2)^2}}\big|_{x=2,y=2,z=1} =$$

$$\sqrt{\frac{a^2+b^2+c^2}{(1+2^2+2^2+1^2)^2}} = \frac{\sqrt{a^2+b^2+c^2}}{10},$$

$$|\mathbf{Y}| = \sqrt{\langle \mathbf{Y}, \mathbf{Y} \rangle} = \sqrt{g_{ik}(x,y,z)Y^i Y^k}\big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik}Y^iY^k}{(1+x^2+y^2+z^2)^2}}\big|_{x=2,y=2,z=1} =$$

$$\sqrt{\frac{p^2+q^2+r^2}{(1+2^2+2^2+1^2)^2}} = \frac{\sqrt{p^2+q^2+r^2}}{10},$$

and

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}||\mathbf{Y}|} = \frac{g_{ik}(x,y,z)X^iY^k\big|_{\mathbf{p}}}{\sqrt{g_{pq}(x,y,z)X^pX^q}\sqrt{g_{rs}(x,y,z)Y^rY^s}} = \frac{\frac{\delta_{ik}X^iY^k}{(1+x^2+y^2+z^2)^2}}{|\mathbf{X}||\mathbf{Y}|}$$

$$= \frac{\frac{ap+bq+cr}{(1+2^2+2^2+1)^2}}{\frac{\sqrt{a^2+b^2+c^2}}{10}\frac{\sqrt{p^2+q^2+r^2}}{10}} == \frac{ap+bq+cr}{\sqrt{a^2+b^2+c^2}\sqrt{p^2+q^2+r^2}}.$$

This example is related with the notion of so called *conformally euclidean metric* (see the next paragraph, 1.2.4).

1.2.4 Conformally Euclidean metric

Let (M,G) be a Riemannian manifold.

Definition We say that metric G is locally conformally Euclidean in a vicinity of the point $\mathbf{p}$ if in a vicinity of this point there exist local coordinates $\{x^i\}$ such that in these coordinates metric has an appearance

$$G = \sigma(x)\delta_{ik}dx^i dx^k = \sigma(x)\left((dx^1)^2 + \cdots + (dx^n)^2\right) , \qquad (1.28)$$

i.e. it is proportional to 'Euclidean metric'. We call coordinates $\{x^i\}$ *conformall* coordinates or *isothermic* coordinates if condition (1.28) holds.

We say that metric is conformally Euclidean if it is locally conformally Euclidean in the vicinity of every point. We say that Riemannian manifold $(M.G)$ is conformally Euclidean if the metric G on it is conformally Euclidean

One can see that if metric is conformally Euclidean in a vicinity of some point $\mathbf{p}$, then the angle between vectors, more precisely the cosinus of the angle between vectors attached at this point (see equation (1.26)) is the same as for Eucldean metric. Indeed, let G be conformally Euclidean metric and let x^i be local coordinates such that the metric has an appearance (1.28) in these coordinates. Let $\mathbf{X}, \mathbf{Y}$ be two non-vanishing vectors $\mathbf{X} = X^m(x)\frac{\partial}{\partial x^m}$, $\mathbf{Y} = Y^m(x)\frac{\partial}{\partial x^m}$ ($|\mathbf{X}| \neq 0, |\mathbf{Y}| \neq 0$) attached at a same point. Then

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y}\rangle}{|\mathbf{X}|\cdot|\mathbf{Y}|} = \frac{g_{ik}X^iY^k}{\sqrt{g_{ik}(x)X^iX^k}\sqrt{g_{ik}(x)Y^iY^k}} =$$

$$\frac{\sigma(x)\delta_{ik}X^iY^k}{\sqrt{\sigma(x)\delta_{ik}(x)X^iX^k}\sqrt{\sigma(x)\delta_{ik}(x)Y^iY^k}} = \frac{\sum_k X^kY^k}{\sqrt{\sum_k(x)X^kX^k}\sqrt{\sum_k Y^kY^k}} . \qquad (1.29)$$

(Note that coefficient σ in equation (1.28) has to be positive.)

Remark One can show that the condition of 'preserving the angles' is not only necessary condition but it is also sufficient condition for metric to be conformally Euclidean (see the problem 1 in Homework 2).

Now consider examples.

First It is instructive to recall the example considered in previous subsection 1.2.3), where Riemannian metric in a vicinity of a point had an appearance (1.27) This is example of Riemannian manifold which is locally conformally Euclidean in a vicinity of a point $\mathbf{p}$.

Another

Example Consider the surface of cylinder with the metric

$$G = a^2 d\varphi^2 + dh^2 \tag{1.30}$$

(see equation (1.24)). In a vicinity of every point one can conisder coordinates $\begin{cases} u = a\varphi \\ v = h \end{cases}$. It is evident that in these coordinates $G = du^2 + dv^2$, i.e. this Riemannian manifold is conformally Euclidean.

Remark. In fact we proved more: for metric of cylinder in coordinates u, v, the coefficient $\sigma(x) \equiv 1$, i,e. in these coordinates metric is not only *locally conformally Euclidean*, but also it is *locally Euclidean*. We will study this question later in details. (see paragraph "Locally Euclidean Riemannian manifold" later).

Later we consider also another important examples.

It is important, that the following Theorem takes a place:

Theorem (Gauss) *Every 2-dimensional Riemannian manifold is locally conformally Euclidean, i.e. for arbitrary 2-dimensional Riemannian manifold, in a vicinity of arbitrary point, there exist coordinates u', v' such that in these coordinates Riemannian metric*

$$G = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2 = \sigma(u',v')\left(du'^2\right) \tag{1.31}$$

We will not prove this theorem footnotethe proof is easy and almost evident for anaylitical manifolds, and it is hard for smooth manifolds, but consider many examples of 2-dimensional Riemannian manifolds, with suitable conformall coordinates.

1.2.5 Length of curves

Let γ: $x^i = x^i(t), (i = 1, \dots, n))$ $(a \le t \le b)$ be a curve on the Riemannian manifold (M, G).

At the every point of the curve the velocity vector (tangent vector) is defined:

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}^n(t) \end{pmatrix} = \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}$$

Remark Note that $\mathbf{v}(t)$ is a vector; check transformation rules:

$$\frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i} = \frac{dx^i(t)}{dt}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial x^{i'}} = \frac{\partial x^{i'}}{\partial x^i}\frac{dx^i(t)}{dt}\frac{\partial}{\partial x^{i'}} = \frac{dx^{i'}(t)}{dt}\frac{\partial}{\partial x^{i'}}\,.$$

The length of velocity vector $\mathbf{v} \in T_xM$ (vector $\mathbf{v}$ is tangent to the manifold M at the point x) equals to

$$|\mathbf{v}|_x = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G\big|_x} = \sqrt{g_{ik}v^iv^k}\big|_x = \sqrt{g_{ik}\frac{dx^i(t)}{dt}\frac{dx^k(t)}{dt}}\Big|_x\,.$$

For an arbitrary curve its length is equal to the integral of the length of velocity vector:

$$L_\gamma = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G\big|_{x(t)}}dt = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt\,. \tag{1.32}$$

Bearing in mind that metric (1.17) defines the length we often write metric in the following form

$$G = ds^2 = g_{ik}dx^idx^k$$

Example 1 Consider 2-dimensional Riemannian manifold with metric

$$||g_{ik}(u,v)|| = \begin{pmatrix} g_{11}(u,v) & g_{12}(u,v) \\ g_{21}(u,v) & g_{22}(u,v) \end{pmatrix}\,.$$

Then

$$G = ds^2 = g_{ik}du^idv^k = g_{11}(u,v)du^2 + 2g_{12}(u,v)dudv + g_{22}(u,v)dv^2\,.$$

The length of the curve γ: $u = u(t), v = v(t)$, where $t_0 \leq t \leq t_1$ according to (1.32) is equal to $L_\gamma = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_{t_0}^{t_1} \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} =$

$$\int_{t_0}^{t_1} \sqrt{g_{11}\left(u\left(t\right), v\left(t\right)\right)u_t^2 + 2g_{12}\left(u\left(t\right), v\left(t\right)\right)u_tv_t + g_{22}\left(u\left(t\right), v\left(t\right)\right)v_t^2}dt\,. \tag{1.33}$$

Example Consider Lobachevsky (hyperbolic) plane. We consider upper-half model of Lobachevsky (hyperbolic) plane:

$$G = \frac{dx^2 + dy^2}{y^2}\,, \quad (y > 0)$$

Consider in Lobachevsky plane the curve C: $\begin{cases} x = x_0 \\ y = t \end{cases}$, $a < t < b$ and claculate its length:

$$L_C = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_a^b \sqrt{g_{ik}(x)\dot{x}^i \dot{x}^k} =$$

$$\int_a^b \sqrt{g_{11}\left(x\left(t\right), y\left(t\right)\right) x_t^2 + 2g_{12}\left(x\left(t\right), y\left(t\right)\right) x_t y_t + g_{22}\left(x\left(t\right), y\left(t\right)\right) y_t^2 dt} =$$

$$\int_a^b \sqrt{\frac{1}{y^2}\left(x_t^2 + y_t^2\right) dt} = \int_a^b \sqrt{\frac{1}{t^2}\left(0+1\right) dt} = \int_a^b \frac{dt}{t} = \left|\log \frac{a}{b}\right| .$$

The length of curves defined by the formula(1.32) obeys the following natural conditions

- It coincides with the usual length in the Euclidean space $\mathbf{E}^n$ ($\mathbf{R}^n$ with standard metric $G = (dx^1)^2 + \cdots + (dx^n)^2$ in Cartesian coordinates). E.g. for 3-dimensional Euclidean space

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t) dt} = \int_a^b \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2 dt}$$

- It does not depend on parameterisation of the curve

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t) dt} = \int_{a'}^{b'} \sqrt{g_{ik}(x(\tau))\dot{x}^i(\tau)\dot{x}^k(\tau) d\tau} \,,$$

  ($x^i(\tau) = x^i(t(\tau))$, $a' \leq \tau \leq b'$ while $a \leq t \leq b$) since under changing of parameterisation

$$\dot{x}^i(\tau) = \frac{dx(t(\tau))}{d\tau} = \frac{dx(t(\tau))}{dt}\frac{dt}{d\tau} = \dot{x}^i(t)\frac{dt}{d\tau} \,.$$

- It *does not depend on coordinates on Riemannian manifold M*

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t) dt} = \int_a^b \sqrt{g_{i'k'}(x'(t))\dot{x}^{i'}(t)\dot{x}^{k'}(t) dt} \,.$$

  This immediately follows from transformation rule (1.71) for Riemannian metric:

$$g_{i'k'}\dot{x}^{i'}(t)\dot{x}^{k'}(t) = g_{ik}\left(\frac{\partial x^i}{\partial x^{i'}(t)}\dot{x}^{i'}(t)\right)\left(\frac{\partial x^k}{\partial x^{k'}}\dot{x}^{k'}(t)\right) g_{ik}\,\dot{x}^i(t)\dot{x}^k(t) \,.$$

- It is additive: length of the sum of two curves is equal to the sum of their lengths. If a curve $\gamma = \gamma_1 + \gamma$, i.e. $\gamma\colon x^i(t), a \leq t \leq b$, $\gamma_1\colon x^i(t), a \leq t \leq c$ and $\gamma_2\colon x^i(t), c \leq t \leq b$ where a point c belongs to the interval (a, b) then $L_\gamma = L_{\gamma^1} + L_{\gamma^2}$.

One can show that formula (1.32) for length is defined uniquely by these conditions. More precisely one can show under some technical conditions one may show that any local additive functional on curves which does not depend on coordinates and parameterisation, and depends on derivatives of curves of order ≤ 1 is equal to (1.32) up to a constant multiplier. To feel the taste of this statement you may do the following exercise:

Exercise Let $A = A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right)$ be a function such that an integral $L = \int A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) dt$ over an arbitrary curve γ in $\mathbf{E}^2$ does not change under reparameterisation of this curve and under an arbitrary isometry, i.e. translation and rotation of the curve. Then one can easy show (show it!) that

$$A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = c\sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2},$$

where c is a constant, i.e. it is a usual length up to a multiplier

1.3 Riemannian structure on the surfaces embedded in Euclidean space

Let M be a surface embedded in Euclidean space. Let G be Riemannian structure on the manifold M.

Let $\mathbf{X}, \mathbf{Y}$ be two vectors tangent to the surface M at a point $\mathbf{p} \in M$. An External Observer calculate this scalar product viewing these two vectors as vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p} \in \mathbf{E}^3$ using scalar product in $\mathbf{E}^3$. An Internal Observer will calculate the scalar product viewing these two vectors as vectors tangent to the surface M using the Riemannian metric G (see the formula (1.38)). Respectively

If L is a curve in M then an External Observer consider this curve as a curve in $\mathbf{E}^3$, calculate the modulus of velocity vector (speed) and the length of the curve using Euclidean scalar product of ambient space. An Internal Observer ("an ant") will define the modulus of the velocity vector and the length of the curve using Riemannian metric.

Definition Let M be a surface embedded in the Euclidean space. We say that metric G_M on the surface is induced by the Euclidean metric if the scalar product of arbitrary two vectors $\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M$ calculated in terms of the metric G equals to Euclidean scalar product of these two vectors:

$$\langle \mathbf{A}, \mathbf{B} \rangle_{G_M} = \langle \mathbf{A}, \mathbf{B} \rangle_{G_{\text{Euclidean}}} \tag{1.34}$$

In other words we say that Riemannian metric on the embedded surface is induced by the Euclidean structure of the ambient space if External and

Internal Observers come to the same results calculating scalar product of vectors tangent to the surface.

In this case modulus of velocity vector (speed) and the length of the curve is the same for External and Internal Observer.

1.3.1 Internal and external observers

Tangent vectors, coordinate tangent vectors

Here we recall basic notions from the course of Geometry which we will need here.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be parameterisation of the surface M embedded in the Euclidean space:

$$\mathbf{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Here as always x, y, z are Cartesian coordinates in $\mathbf{E}^3$.

Let $\mathbf{p}$ be an arbitrary point on the surface M. Consider the plane formed by the vectors which are adjusted to the point $\mathbf{p}$ and tangent to the surface M. We call this plane *plane tangent to M at the point* $\mathbf{p}$ and denote it by $T_{\mathbf{p}}M$.

For a point $\mathbf{p} \in M$ one can consider a basis in the tangent plane T_pM adjusted to the parameters u, v.

Tangent basis vectors at any point (u, v) are

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u, v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} = \frac{\partial x(u, v)}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial u} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial u} \frac{\partial}{\partial z}$$

and

$$\mathbf{r}_v = \frac{\partial \mathbf{r}(u, v)}{\partial v} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{pmatrix} = \frac{\partial x(u, v)}{\partial v} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial v} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial v} \frac{\partial}{\partial z}$$

Definition We call basis vectors $\mathbf{r}_u, \mathbf{r}_v$ adjusted to parameters (coordinates) u, v *coordinate basis vectors*

Every vector $\mathbf{X} \in T_pM$ can be expanded over the basis of coordinate basis vectors:

$$\mathbf{X} = X_u \mathbf{r}_u + X_v \mathbf{r}_v,$$

where X_u, X_v are coefficients, components of the vector $\mathbf{X}$.

Internal Observer views the basis vector $\mathbf{r}_u \in T_pM$ as the vector ∂_u. Why? The vector $\mathbf{r}_u$ attached at the point $\mathbf{p}$ is a velocity vector for the curve $\gamma_{\mathbf{r}_u}(t)\colon \begin{cases} u = u_0 + t \\ v = v_0 \end{cases}$ starting at the point $\mathbf{p}$ ((u_0, v_0) are coordinates of the point $\mathbf{p}$). If $f = f(u, v)$ is a function on the surface M, then one can see that directional derivative of this function along a vector $\mathbf{r}_u$ is defined by $\frac{\partial}{\partial u}$:

$$\partial_u f(u, v)\big|_{\mathbf{p}} = \frac{d}{dt} f\left(\gamma_{\mathbf{r}_u}(t)\right) = \frac{d}{dt} f\left(u_0 + t, v_0\right) .$$

Respectively the basis vector $\mathbf{r}_v \in T_pM$ for an Internal Observer, is velocity vector for the curve $\gamma_{\mathbf{r}_v}(t)\colon \begin{cases} u = u_0 \\ v = v_0 + t \end{cases}$ starting at the point $\mathbf{p}$ and Internal Observer denotes this vector ∂_v:

$$\partial_v f(u, v)\big|_{\mathbf{p}} = \frac{d}{dt} f\left(\gamma_{\mathbf{r}_v}(t)\right) = \frac{d}{dt} f\left(u_0, v_0 + t\right) .$$

For an arbitrary vector $\mathbf{X}$ which is tangent to surface M at the point $\mathbf{p}$, ($\mathbf{X} \in T_pM$)

External observer	Internal observer
$\mathbf{X} = a\mathbf{r}_u + b\mathbf{r}_v$	$\mathbf{X} = a\partial_u + b\partial_v$

Example Consider sphere of radius R in $\mathbf{E}^3$, $x^2 + y^2 + z^2 = R^2$. In spherical coordinates

$$\mathbf{E}^3 \ni \mathbf{r} = \mathbf{r}(\theta, \varphi) \begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ x = R \cos\theta \end{cases} ,$$

these coordinates are well-defined for $0 < \theta < \frac{\pi}{2}$ and $0 < \varphi < 2\pi$. For coordinate basis vectors $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ we have:

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\theta, \varphi)}{\partial \theta} = \frac{\partial}{\partial \theta} \begin{pmatrix} x(\theta, \varphi) \\ y(\theta, \varphi) \\ z(\theta, \varphi) \end{pmatrix} = \frac{\partial}{\partial \theta} \begin{pmatrix} R \sin\theta \cos\varphi \\ R \sin\theta \sin\varphi \\ R \cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} R \cos\theta \cos\varphi \\ R \cos\theta \sin\varphi \\ -R \sin\theta \end{pmatrix} = R \cos\theta \cos\varphi \frac{\partial}{\partial x} + R \cos\theta \sin\varphi \frac{\partial}{\partial y} - R \sin\theta \frac{\partial}{\partial z} ,$$

and respectively

$$\mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\theta} = \frac{\partial}{\partial\varphi}\begin{pmatrix} x(\theta,\varphi)\\ y(\theta,\varphi)\\ z(\theta,\varphi)\end{pmatrix} = \frac{\partial}{\partial\varphi}\begin{pmatrix} R\sin\theta\cos\varphi\\ R\sin\theta\sin\varphi\\ R\cos\theta\end{pmatrix} =$$

$$\begin{pmatrix} -R\sin\theta\sin\varphi\\ R\sin\theta\cos\varphi\\ 0\end{pmatrix} = -R\sin\theta\sin\varphi\frac{\partial}{\partial x} + R\sin\theta\cos\varphi\frac{\partial}{\partial y}\,. \tag{1.35}$$

Here is a table how observers look at the objects on sphere:

	INTERNAL OBSERVER	EXTERNAL OBSERVER
point on S^2	2 coordinates θ,φ	3 coordinates $\mathbf{r}=\mathbf{r}(\theta,\varphi)$
curve on S^2	$\theta(t),\varphi(t)$	$\mathbf{r}(t)=\mathbf{r}(\theta(t,\varphi(t)))$
coordinate tangent vectors to S^2	$\frac{\partial}{\partial\theta}\,,\frac{\partial}{\partial\varphi}$	$\mathbf{r}_\theta\,,\quad \mathbf{r}_\varphi$
tangent vector to S^2	$a\frac{\partial}{\partial\theta}+b\frac{\partial}{\partial\varphi}$	$A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}+C\frac{\partial}{\partial z}=a\mathbf{r}_\theta+b\mathbf{r}_\varphi$

Explicit formulae for induced Riemannian metric (First Quadratic form)

Now we are ready to write down the explicit formulae for the Riemannian metric on the surface induced by metric (scalar product) in ambient Euclidean space (see the Definition (1.34). We will return to induced metric again in next paragraph1.3.2.

Let $M\colon \mathbf{r}=\mathbf{r}(u,v)$ be a surface embedded in $\mathbf{E}^3$.

The formula (1.34) means that scalar products of basic vectors $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$ has to be the same calculated on the surface or in the ambient space, i.e. calculated by Internal observer, or by External observer. For example scalar product $\langle\partial_u,\partial_v\rangle_M = g_{uv}$ calculated by the Internal Observer is the same as a scalar product $\langle\mathbf{r}_u,\mathbf{r}_v\rangle_{\mathbf{E}^3}$ calculated by the External Observer, scalar product $\langle\partial_v,\partial_v\rangle_M = g_{uv}$ calculated by the Internal Observer is the same as a scalar product $\langle\mathbf{r}_v,\mathbf{r}_v\rangle_{\mathbf{E}^3}$ calculated by the External Observer and so on:

$$G = \begin{pmatrix} g_{uu} & g_{uv}\\ g_{vu} & g_{vv}\end{pmatrix} = \begin{pmatrix} \langle\partial_u,\partial_u\rangle & \langle\partial_u,\partial_v\rangle\\ \langle\partial_v,\partial_u\rangle & \langle\partial_v,\partial_v\rangle\end{pmatrix} = \begin{pmatrix} \langle\mathbf{r}_u,\mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle\mathbf{r}_u,\mathbf{r}_v\rangle_{\mathbf{E}^3}\\ \langle\mathbf{r}_v,\mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle\mathbf{r}_v,\mathbf{r}_v\rangle_{\mathbf{E}^3}\end{pmatrix} \tag{1.36}$$

where as usual we denote by $\langle\ ,\ \rangle_{\mathbf{E}^3}$ the scalar product in the ambient Euclidean space.

Remark It is convenient sometimes to denote parameters (u, v) as (u^1, u^2) or u^α $(\alpha = 1, 2)$ and to write $\mathbf{r} = \mathbf{r}(u^1, u^2)$ or $\mathbf{r} = \mathbf{r}(u^\alpha)$ $(\alpha = 1, 2)$ instead $\mathbf{r} = \mathbf{r}(u, v)$

In these notations:

$$G_M = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} \end{pmatrix}, \quad g_{\alpha\beta} = \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle ,$$

$$G_M = g_{\alpha\beta} du^\alpha du^\beta = g_{11} du^2 + 2 g_{12} du dv + g_{22} dv^2 \tag{1.37}$$

where (,) is a scalar product in Euclidean space.

The formula (1.37) is the formula for induced Riemannian metric on the surface $\mathbf{r} = \mathbf{r}(u, v)$ [2].

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane $T_p C$ then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$:

$$(\mathbf{X}, \mathbf{Y}) = (X^1 \mathbf{r}_1 + X^2 \mathbf{r}_2, Y^1 \mathbf{r}_1 + Y^2 \mathbf{r}_2) = \tag{1.38}$$

$$X^1 (\mathbf{r}_1, \mathbf{r}_1) Y^1 + X^1 (\mathbf{r}_1, \mathbf{r}_2) Y^2 + X^2 (\mathbf{r}_2, \mathbf{r}_1) Y^1 + X^2 (\mathbf{r}_2, \mathbf{r}_2) Y^2 =$$
$$X^\alpha (\mathbf{r}_\alpha, \mathbf{r}_\beta) Y^\beta = X^\alpha g_{\alpha\beta} Y^\beta = G(\mathbf{X}, \mathbf{Y})$$

1.3.2 Formulae for induced metric

We obtained (1.37) from equation (1.34).

We can do these calculations in a little bit other way.

The Riemannian structure of Euclidean space— standard Euclidean metric in Euclidean coordinates is given by

$$G_{\mathbf{E}^3} = (dx)^2 + (dy)^2 + (dz)^2 . \tag{1.39}$$

Then the induced metric (1.34) on the surface M defined by equation $\mathbf{r} = \mathbf{r}(u, v)$ is equal to

$$G_M = G_{\mathbf{E}^3}\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \left((dx)^2 + (dy)^2 + (dz)^2\right)\big|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta} du^\alpha du^\beta \tag{1.40}$$

i.e. $((dx)^2 + (dy)^2 + (dz)^2)\big|_{\mathbf{r}=\mathbf{r}(u,v)} =$

$$\left(\frac{\partial x(u,v)}{\partial u} du + \frac{\partial x(u,v)}{\partial v} dv\right)^2 + \left(\frac{\partial y(u,v)}{\partial u} du + \frac{\partial y(u,v)}{\partial v} dv\right)^2 + \left(\frac{\partial z(u,v)}{\partial u} du + \frac{\partial z(u,v)}{\partial v} dv\right)^2 =$$

[2] it is called sometimes First Quadratic Form of this surface.

$$(x_u^2+y_u^2+z_u^2)du^2+2(x_ux_v+y_uy_v+z_uz_v)dudv+(x_v^2+y_v^2+z_v^2)dv^2$$

We see that

$$G_M = g_{\alpha\beta}du^\alpha du^\beta = g_{11}du^2+2g_{12}dudv+g_{22}dv^2\,, \tag{1.41}$$

where for matrix $||g_{\alpha\beta}||$, $(\alpha,\beta=1,2)$,

$$||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} =$$

$$\begin{pmatrix} (x_u^2+y_u^2+z_u^2) & (x_ux_v+y_uy_v+z_uz_v) \\ (x_ux_v+y_uy_v+z_uz_v) & (x_v^2+y_v^2+z_v^2) \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v\rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_v, \mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v\rangle_{\mathbf{E}^3} \end{pmatrix}. \tag{1.42}$$

We come to same formula (1.37).

Example Consider again sphere of radius R in $\mathbf{E}^3$, $x^2+y^2+z^2=R^2$ in stereographic coordinates. We calculated coordinate tangent vectors to this sphere in (1.35). Now calculate induced Riemannian metric:

$$G_{S^2} = \left(dx^2+dy^2+dz^2\right)\big|_{x=R\sin\theta\cos\varphi, y=R\sin\theta\sin\varphi, x=R\cos\theta\cos\theta} =$$

$$[d(R\sin\theta\cos\varphi)]^2+[d(R\sin\theta\sin\varphi)]^2+[d(R\cos\theta)]^2 =$$

$$[R\cos\theta\cos\varphi d\theta - R\sin\theta\sin\varphi d\varphi]^2+[R\cos\theta\sin\varphi d\theta + R\sin\theta\cos\varphi d\varphi]^2+[-R\sin\theta d\theta]^2 =$$

$$(R^2\sin^2\theta\sin^2\varphi+R^2\sin^2\theta\cos^2\varphi)d\varphi^2+(R^2\cos^2\theta\cos^2\varphi+R^2\sin^2\theta)d\theta^2 = R^2d\theta^2+R^2\sin^2\theta d\varphi^2\,.$$

We see that

$$G_{S^2} = R^2d\theta^2+R^2\sin^2\theta d\varphi^2\,, ||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}. \tag{1.43}$$

Remark Sometimes it is useful to use the following "condensed" notations. We denote Cartesian coordinates (x,y,z) of Euclidean space by x^i, $(i=1,2,3)$. Let surface M be given in local parameterisation $x^i=x^i(u^\alpha)$. Riemannian metric of Euclidean space (1.39) has appearance

$$G_{\mathbf{E}} = dx^i\delta_{ik}dx^k\,. \tag{1.44}$$

and calculations (1.40) —(1.42) for induced metric (1.40) has appearance

$$G_M = dx^i\delta_{ik}dx^k\big|_{x^i=x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha}\delta_{ik}\frac{\partial x^k(u)}{\partial u^\beta}du^\alpha du^\beta = g_{\alpha\beta}(u)du^\alpha du^\beta \tag{1.45}$$

(See also remark above before equation (1.37)). One can rewrite (1.45) in the following way:

$$g_{\alpha\beta} = \frac{\partial x^i(u)}{\partial u^\alpha}\delta_{ij}\frac{\partial x^j(u)}{\partial u^\beta}\,. \quad (\alpha,\beta = 1,2,)\,.$$

It is instructive to come to this equation straightforwardly from equation (1.34) and definition (1.17). We have that due to (1.34)

$$g_{\alpha\beta} = g_{\pi\rho}dx^\pi dx^\rho\left(\frac{\partial}{\partial u^\alpha},\frac{\partial}{\partial u^\beta}\right) = G_M\left(\frac{\partial}{\partial u^\alpha},\frac{\partial}{\partial u^\beta}\right) = G_{\mathbf{E}^3}\left(\mathbf{r}_\alpha,\mathbf{r}_\beta\right) =$$

$$\delta_{pq}dx^p dx^q\left(\frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i},\frac{\partial x^j}{\partial u^\beta}\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial u^\alpha}\left[\delta_{pq}dx^p dx^q\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\right]\frac{\partial x^j}{\partial u^\beta} = \frac{\partial x^i}{\partial u^\alpha}\delta_{ij}\frac{\partial x^j}{\partial u^\beta}\,.$$

Representation (1.45) in condenced notations is very useful. It is easy to see that this formula works for arbitrary dimensions, i.e. if we have m-dimensional manifold embedded in n-dimensional Euclidean space. We just have to suppose that in this case $i = 1,\dots,n$ and $\alpha = 1,\dots,m$; manifold is given by parameterisation $x^i = x^i(u^\alpha)$ $(\alpha = 1,\dots,m)$. Moreover in the case if manifold is embedded not in Euclidean space but in an arbitrary Riemannian space then one can see that we come to the induced metric

$$G_M = dx^i g_{ik}\left((x(u))\right)dx^k\big|_{x^i=x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha}g_{ik}\left((x(u))\right)\frac{\partial x^k(u)}{\partial u^\beta}du^\alpha du^\beta = g_{\alpha\beta}(x(u))du^\alpha du^\beta$$

Check explicitly again that length of the tangent vectors and curves on the surface calculating by External observer (i.e. using Euclidean metric (1.39)) *is the same* as calculating by Internal Observer, ant (i.e. using the induced Riemannian metric (1.37), (1.41)). Let $\mathbf{X} = X^\alpha\mathbf{r}_\alpha = a\mathbf{r}_u + b\mathbf{r}_v$ be a vector tangent to the surface M. The square of the length $|\mathbf{X}|$ of this vector calculated by External observer (he calculates using the scalar product in $\mathbf{E}^3$) equals to

$$|\mathbf{X}|^2 = \langle\mathbf{X},\mathbf{X}\rangle = \langle a\mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v\rangle = a^2\langle\mathbf{r}_u,\mathbf{r}_u\rangle + 2ab\langle\mathbf{r}_u,\mathbf{r}_v\rangle + b^2\langle\mathbf{r}_v,\mathbf{r}_v\rangle \tag{1.46}$$

where $\langle\,,\,\rangle$ is a scalar product in $\mathbf{E}^3$. The internal observer will calculate the length using Riemannian metric (1.37) (1.41):

$$G(\mathbf{X},\mathbf{X}) = \begin{pmatrix}a, & b\end{pmatrix}\cdot\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & G_{22}\end{pmatrix}\cdot\begin{pmatrix}a\\ b\end{pmatrix} = g_{11}a^2 + 2g_{12}ab + g_{22}b^2 \tag{1.47}$$

External observer (person living in ambient space $\mathbf{E}^3$*) calculates the length of the tangent vector using formula* (1.46)*. An ant living on the surface* *calculates length of this vector in internal coordinates using formula* (1.47). *External observer deals with external coordinates of the vector, ant on the surface with internal coordinates. They come to the same answer.*

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \le t \le b$ be a curve on the surface.

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to

$$\mathbf{v} = \mathbf{X} = \xi \mathbf{r}_u + \eta \mathbf{r}_v \text{where } \xi = u_t, \eta = v_t\colon \quad \mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t \mathbf{r}_u + v_t \mathbf{r}_v\,.$$

The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{\langle \mathbf{v}(t), \mathbf{v}(t)\rangle_{\mathbf{E}^3}} dt = \int_a^b \sqrt{\langle u_t \mathbf{r}_u + v_t \mathbf{r}_v, u_t \mathbf{r}_u + v_t \mathbf{r}_v\rangle_{\mathbf{E}^3}} dt = \tag{1.48}$$

$$\int_a^b \sqrt{\langle \mathbf{r}_u, \mathbf{r}_u\rangle_{\mathbf{E}^3} u_t^2 + 2\langle \mathbf{r}_u, \mathbf{r}_v\rangle_{\mathbf{E}^3} u_t v_t + \langle \mathbf{r}_v, \mathbf{r}_v\rangle_{\mathbf{E}^3} v_t^2} d\tau =$$

$$\int_a^b \sqrt{g_{11} u_t^2 + 2 g_{12} u_t v_t + g_{22} v_t^2} dt \tag{1.49}$$

An external observer will calculate the length of the curve using (1.48). *An ant living on the surface calculate length of the curve using* (1.49) *using Riemannian metric on the surface. They will come to the same answer.*

1.3.3 Induced Riemannian metrics. Examples.

We consider already an example of induced Riemannian metric on sphere in spherical coordinates. Now we consider here other examples of induced Riemannian metric on some surfaces in $\mathbf{E}^3$. using calculations for tangent vectors (see (1.37)) or explicitly in terms of differentials (see (1.40) and (1.41)).

First of all consider the general case when a surface M is defined by the equation $z - F(x, y) = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(u, v)\colon \quad \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \tag{1.50}$$

Then coordinate tangent vectors $\mathbf{r}_u, \mathbf{r}_v$ are

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} \tag{1.51}$$

,

$$(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2, \quad (\mathbf{r}_u, \mathbf{r}_v) = F_u F_v, \quad (\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$$

and induced Riemannian metric (first quadratic form) (1.37) is equal to

$$||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix} \tag{1.52}$$

$$G_M = ds^2 = (1 + F_u^2)du^2 + 2F_u F_v du dv + (1 + F_v^2)dv^2 \tag{1.53}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on C $(a \leq t \leq b)$ can be calculated by the formula:

$$L = \int \int_a^b \sqrt{(1 + F_u^2)u_t^2 + 2F_u F_v u_t v_t + (1 + F_v)^2 v_t^2} dt$$

One can calculate (1.53) explicitly using (1.40):

$$G_M = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=u, y=v, z=F(u,v)} = (du)^2 + (dv)^2 + (F_u du + F_v dv)^2 =$$

$$= (1 + F_u^2)du^2 + 2F_u F_v du dv + (1 + F_v^2)dv^2 \,. \tag{1.54}$$

Cylinder

Cylinder is given by the equation $x^2 + y^2 = a^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} \tag{1.55}$$

We have $G_{cylinder} = \iota^* G_{\mathbf{E}^3} = (dx^2 + dy^2 + dz^2)\big|_{x=a\cos\varphi, y=a\sin\varphi, z=h} =$

$$= (-a\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2 \tag{1.56}$$

The same formula in terms of scalar product of tangent vectors:

$$\text{coordinate basis vectors } \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix} \tag{1.57}$$

,

$$(\mathbf{r}_h, \mathbf{r}_h) = 1, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2$$

and

$$||g_{\alpha\beta}|| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix},$$

$$G = dh^2 + a^2 d\varphi^2 \tag{1.58}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cylinder $(a \leq t \leq b)$ can be calculated by the formula:

$$L = \int_a^b \sqrt{h_t^2 + a^2\varphi_t} dt \tag{1.59}$$

Cone

Cone is given by the equation $x^2 + y^2 - k^2 z^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \tag{1.60}$$

Calculate induced Riemannian metric:
We have

$$G_{conus} = \iota^* G_{\mathbf{E}^3} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} =$$

$$(k\cos\varphi dh - kh\sin\varphi d\varphi)^2 + (k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2$$

$$G_{conus} = k^2h^2d\varphi^2 + (1+k^2)dh^2,\ ||g_{\alpha\beta}|| = \begin{pmatrix} 1+k^2 & 0 \\ 0 & k^2h^2 \end{pmatrix} \tag{1.61}$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cone $(a \leq t \leq b)$ can be calculated by the formula:

$$L = \int_a^b \sqrt{(1+k^2)h_t^2 + k^2h^2\varphi_t^2} dt \tag{1.62}$$

Circle `(again)`

Circle of radius R is given by the equation $x^2 + y^2 = R^2$. Consider standard parameterisation φ of this surface:

$$\mathbf{r}(\varphi)\colon \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$$

Calculate induced Riemannian metric (first quadratic form)

$$G_{S^1} = \iota^* G_{\mathbf{E}^3} = \left(dx^2 + dy^2\right)\big|_{x=R\cos\varphi, y=R\sin\varphi} =$$

$$(-R\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 = (R^2\cos^2\varphi + R^2\sin^2\varphi)d\varphi^2 = R^2 d\varphi^2\,.$$

One can consider stereographic coordinates on the circle (see Example in the subsection 1.1) A point $x, y\colon x^2 + y^2 = R^2$ has stereographic coordinate t if points $(0, 1)$ (north pole), the point (x, y) and the point $(t, 0)$ belong to the same line, i.e. $\frac{x}{t} = \frac{R-y}{R}$, i.e.

$$t = \frac{Rx}{R-y}, \qquad \begin{cases} x = \frac{2tR^2}{R^2+t^2} \\ y = \frac{t^2-R^2}{t^2+R^2}R \end{cases} \quad . \quad \text{since } x^2 + y^2 = R^2\,.$$

Induced metric in coordinate t is

$$G = (dx^2 + dy^2)\big|_{x=x(t), y=y(t)} = \left(d\left(\frac{2tR^2}{R^2+t^2}\right)\right)^2 + \left(d\left(\frac{t^2-R^2}{R^2+t^2}R\right)\right)^2 =$$

$$\left(\frac{2R^2dt}{R^2+t^2} - \frac{4t^2R^2dt}{(R^2+t^2)^2}\right)^2 + \left(-\frac{4R^2tdt}{(t^2+R^2)^2}\right)^2 = \frac{4R^4dt^2}{(R^2+t^2)^2}\,. \tag{1.63}$$

(See for detail Homework 2[3].

Remark Stereographic coordinates very often are preferable since they define birational equivalence between circle and line.

Sphere `(again...)`

Sphere of radius R is given by the equation $x^2 + y^2 + z^2 = R^2$. Consider first stereographic coordinates

$$\mathbf{r}(\theta, \varphi)\colon \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{1.64}$$

[3]One can also obtain this formula in a very beautiful way using inversion (see Appendices)

We already calculated the coordinate basis in (1.35) and we calculated induced Riemannian metric in (1.43):

$$, \qquad G_{S^2} = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2 \,, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{1.65}$$

One comes to the same answer calculating scalar product of coordinate tangent vectors:

coordinate tangent vectors are $\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix} \,, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$

,

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = R^2, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2\sin^2\theta$$

and

$$||g|| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} =$$

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}, \quad G_{S^2} = ds^2 = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ on the sphere of the radius a ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b R\sqrt{\theta_t^2 + \sin^2\theta\cdot\varphi_t^2}dt \tag{1.66}$$

One can consider on sphere as well as on a circle stereographic coordinates:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases} \,, \qquad \begin{cases} x = \frac{2uR^2}{R^2+u^2+v^2} \\ y = \frac{2vR^2}{R^2+u^2+v^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2}R \end{cases} \tag{1.67}$$

In these coordinates Riemannian metric is

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} =$$

$$\left(d\left(\frac{2uR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{2vR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(1 - \frac{2R^2}{R^2+u^2+v^2}\right)R\right)^2 =$$

$$= \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2} \,. \qquad (1.68)$$

(See for detail Homework 2[4].

Notice that we showed that metric on sphere is conformally Euclidean.

Saddle (paraboloid)

Consider paraboloid $z = x^2 - y^2$. It can be rewritten as $z = axy$ and it is called sometimes "saddle" (rotation on the angle $\varphi = \pi/4$ transforms $z = x^2 - y^2$ onto $z = 2xy$.) We considered this example in homework 3 Paraboloid and saddle they are ruled surfaces which are formed by lines.

Examples of other quadratic surfaces see in in Appendix.

1.4 Isometries of Riemannian manifolds.

1.4.1 Riemannian metric induced by map

Let M be a manifold, and let $(N, G(N))$ be a Riemannian manifold. Let F be a map from M to N,

$$F\colon \qquad M \longrightarrow N$$

We do not suppose that F is diffeomorphism, we even do not suppose that manifolds M and N have the same dimension[5].

We can define F^*G, a *Riemannian metric on manifold M induced by the map $F\colon M \to N$.*

Describe the metric F^*G on M in local coordinates.

Let x^a, $(a = 1, \ldots, m)$ be local coordinates on m-dimensional manifold M in a vicinity of some point $\mathbf{p}_M$ on M. Consider a point $\mathbf{p}_N = F(\mathbf{p}_M)$ on manifold N and let y^i, $(i = 1, \ldots, n)$ be local coordinates on n-dimensional manifold N in a vicinity of point $\mathbf{p}_N$. If in local coordinates y^i, Riemannian metric $G^{(N)}$ on N has appearance

$$G^{(N)} = g^{(N)}_{ij}(y)dy^i dy^j \,,$$

then in local coordinates x^a, Riemannian metric $F^*(G^{(N)})$ on M has appearance

$$F^*(G^{(N)}) = g^{(M)}_{ab}(x)dx^a dx^b = dx^a \frac{\partial y^i(x)}{\partial x^a} g^{(N)}_{ij}(y(x)) \frac{\partial y^j(x)}{\partial x^b} dx^b \,,$$

[4]Another beautiful deduction of this formula see in Appendices (Inversion)

[5]We just suppose that F is differentiable map, i.e. local expressions for F are smooth functions

i.e.

$$g_{ab}^{(M)}(x) = \frac{\partial y^i(x)}{\partial x^a} g_{ij}^{(N)}(y(x)) \frac{\partial y^j(x)}{\partial x^b}\,, \tag{1.69}$$

where $y^i = y^i(x^a)$ is expression of map F in local coordinates x^a and y^i,

Remark The metric $F^*(G^{(N)})$ on M is called *pull-back* of metric $G^{(N)}$ under the map $F\colon\ M \to N$.

Example The induced metric on surfaces in $\mathbf{E}^3$ is a special example of this general construction. Indeed embedding $\iota\colon\ M \to \mathbf{E}^3$ is a map from points of 2-dimensional manifold M to points of 3-dimensional Euclidean space $\mathbf{E}^3$. Applying (1.69) to this map we come to formulae (1.41) for induced metric:

$$G_M = \iota^* G_{\mathbf{E}^3} = \iota^*(dx^2 + dy^2 + dz^2) =$$

$$G_{\mathbf{E}^3}\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \left((dx)^2 + (dy)^2 + (dz)^2\right)\big|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta} du^\alpha du^\beta$$

(or another manifestation of this formula, equation (1.45)).

1.4.2 Diffeomorphism, which is an isometry

Let $(M_1, G_{(1)})$, $(M_2, G_{(2)})$ be two Riemannian manifolds— manifolds equipped with Riemannian metric $G_{(1)}$ and $G_{(2)}$ respectively.

Loosely speaking isometry is the diffeomorphism of Riemannian manifolds which preserves the distance.

Definition Let F be a diffeomorphims (one-one smooth map with smooth inverse) of manifold M_1 on manifold M_2.

We say that diffeomorphism F is an isometry of Riemannian manifolds $(M_1, G_{(1)})$ and $(M_2, G_{(2)})$ if it preserves the metrics, i.e. $G_{(1)}$ is pull-back of $G_{(2)}$:

$$F^* G_{(2)} = G_{(1)}\,. \tag{1.70}$$

According (1.69) this means that

$$F^*\left(g_{(2)ab}(y) dy^a dy^b\right) = g_{(2)ab}(y) dy^a dy^b\big|_{y=y(x)} =$$

$$g_{(2)ab}(y(x)) \frac{\partial y^a(x)}{\partial x^i} dx^i \frac{\partial y^b(x)}{\partial x^k} dx^k = g_{(1)ik}(x) dx^i dx^k\,,$$

i.e.

$$g_{(1)ik}(x) = \frac{\partial y^a(x)}{\partial x^i} g_{(2)ab}(y(x)) \frac{\partial y^b(x)}{\partial x^k}\,, \tag{1.71}$$

where $y^a = y^a(x)$ is local expression for diffeomorphism F. We say that diffeomorphism F is *isometry* of Riemanian manifolds $(M_1, G_{(1)})$ and $M_2, G_{(2)}$. The difference of this equation with equation (1.69) is that F in (1.69) was just a differentiable map, which is not a diffeomorphism. In (1.71) diffeomorphism F establishes one-one correspondence between local coordinates on manifolds M_1 and M_2. The left hand side of equation (1.71) can be considered as a local expression of metric $G_{(2)}$ in coordinates x^i on M_2 and the right hand side of this equation is local expression of metric $G_{(1)}$ in coordinates x^i on M_1. Diffeomorphism F identifies manifolds M_1 and M_2 and it can be considered as changing of coordinates.

Example Consider surface of cylinder C, $x^2+y^2 = a^2$ in $\mathbf{E}^3$ with induced Riemannian metric $G_C = a^2 d\varphi^2 + dh^2$ (see equations (1.55) and (1.56)). If we remove the line l: $x = a, y = 0$ from the cylinder surface C we come to surface $C' = C \backslash l$. Consider a map F of this surface in Euclidean space E^2 with Cartesian coordinates u, v (with standard Euclidean metric $G_{Eucl} = du^2 + dv^2$):

$$F\colon \qquad \begin{cases} u = a\varphi \\ v = h \end{cases} \qquad 0 < \varphi < 2\pi\,. \tag{1.72}$$

One can see that F is the diffeomorphism of C' on the domain $0 < u < 2\pi a$ in $\mathbf{E}^2$ and this diffeomorphism is an isometry: it transforms the metric G_{Eucl} on Euclidean space in metric G_C on cylinder, i.e. pull-back condition (1.70) is obeyed:

$$F^* G_{Eucl} = F^*\left(du^2 + dv^2\right) = \left(du^2 + dv^2\right)\big|_{u=a\varphi, v=h} = a^2 d\varphi^2 + dh^2 = G_1\,.$$

We see that cylinder surface with removed line is isometric to domain in $\mathbf{E}^2$ and the map F establishes this isometry.

Remark Notice that if F is diffeomorphism of manifold M_1 on a Riemannian manifold $(M_2, G_{(2)}$, then it defines Riemannian structure, the pull-back $G_1 = F^*(G(2))$ on M_1, and F is isometry of Riemannian manifold $(M_1, G_{(1)})$ on Riemannian manifold $(M_2, G_{(2)})$.

1.4.3 Isometries of Riemannian manifold on itself

Definition Let (M, G) be a Riemannian manifold. We say that a diffeomorphism F is an isometry of Riemannian manifold on itself if it preserves the metric, i.e. $F^*G = G$. In local coordinates this means that

$$g_{ik}(x) = g_{pq}(x'(x))\frac{\partial x^p(x')}{\partial x^i}\frac{\partial x^q(x')}{\partial x^k}\,, \tag{1.73}$$

where $x' = x'(x)$ is a local expression for diffeomorphism F. **Example** Let $\mathbf{E}^2$ be Euclidean plane with metric $dx^2 + dy^2$ in Cartesian coordinates x, y. Consider the transformation

$$\begin{cases} x' = p + ax + by \\ y' = q + cx + dy \end{cases}$$

is isometry if and only if the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an orthogonal matrix, i.e. if the trasformation above is combination of translation, rotation and reflection.

Another example **Example** Consider Lobachevsky (hyperbolic) plane: an upper half-plain ($y > 0$) in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2},$$

One can see that the map

$$\begin{cases} x = \lambda x' \\ y = \lambda y' \end{cases}, (\lambda > 0)$$

($\lambda > 0$) is an isometry of the Lobachevsky plane on itself. Are there other isometries? Yes there are (See the disucssion of these questions in Homeworks.)

1.4.4 Locally Euclidean Riemannian manifolds

It is useful to formulate the local isometry condition between Riemannian manifold and Euclidean space. A neighbourhood of every point of n-dimensional manifold is diffeomorphic to $\mathbf{R}^n$. Let as usual $\mathbf{E}^n$ be n-dimensional Euclidean space, i.e. $\mathbf{R}^n$ with standard Riemannian metric $G = dx^i \delta_{ik} dx^k = (dx^1)^2 + \cdots + (dx^n)^2$ in Cartesian coordinates $(x^1, \ldots, x^n)$.

Definition We say that n-dimensional Riemannian manifold (M, G) is locally isometric to Euclidean space $\mathbf{E}^n$, i.e. it is locally Euclidean Riemannian manifold, if for every point $\mathbf{p} \in M$ there exists an open neighboorhood D (domain) containing this point, $\mathbf{p} \in D$ such that D is isometric to a domain in Euclidean plane. In other words in a vicinity of every point $\mathbf{p}$ there exist local coordinates $u^1, \dots, u^n$ such that Riemannian metric G in these coordinates has an appearance

$$G = du^i \delta_{ik} du^k = (du^1)^2 + \cdots + (du^n)^2 \,. \tag{1.74}$$

The coordinates $(u^1, \dots, u^n)$ are called *locally Euclidean coordinates.*

Consider examples.

Example Consider again cylinder surface..

We know that cylinder is not diffeomorphic to plane (cylinder surface is $S^1 \times \mathbf{R}$, $\mathbf{E}^2 = \mathbf{R} \times \mathbf{R}$, and circle is not diffeomorphic to line). In the previous subsection we cutted the line from cylindre. Thus we came to surface diffeomorphic to plane. We established that this surface is isometric to Euclidean plane. (See equation (1.72) and considerations above.) Local isometry of cylinder to the Euclidean plane, i.e. the fact that it is locally Euclidean Riemannian surface immediately follows from the fact that under changing of local coordinates $u = a\varphi, v = h$ in equation (1.72), the standard Euclidean metric $du^2 + dv^2$ transforms to the metric $G_{cylinder} = a^2 d\varphi^2 + dh^2$ on cylinder.

Remark Strictly speaking we consider all the points except the points on the cutted line (with coordinate $\varphi - 0$). On the other hand for the points on cuttng line we can consider instead coordinate φ another coordinate $\varphi' = \varphi - \pi$, $-\pi < \varphi' < \pi$, and we will come to the same answer. In this case the cutted line will be the line $\varphi / = \pi$.

Example Now show that cone is locally Euclidean Riemannian surface, i.e, it is locally isometric to the Euclidean plane. This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$. Recall calculations of the metric on cone in coordinates h, φ where

$$\mathbf{r}(h, \varphi) \colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} ,$$

$x^2 + y^2 - k^2 z^2 = k^2 h^2 \cos^2\varphi + k^2 h^2 \sin^2\varphi - k^2 h^2 = k^2 h^2 - k^2 h^2 = 0$. We have that metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2 + dy^2 + dz^2$ is equal to

$$G_c = \left(dx^2 + dy^2 + dz^2\right)\big|_{x = kh\cos\varphi, y = kh\sin\varphi, z = h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$

$$(k \sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2\,.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h \cos\beta\varphi \\ v = \alpha h \sin\beta\varphi \end{cases}$, where α, β are parameters. We come to $du^2 + dv^2 =$

$$(\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2.$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = \sqrt{k^2+1}$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2 + dv^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane $\blacksquare$

Of course these coordinates are local.— Cone and plane are not homeomorphic, thus they are not globally isometric.

Example and counterexample

Consider domain D in Euclidean plane with two metrics:

$$G_{(1)} = du^2 + \sin^2 v dv^2\,, \quad \text{and} \quad G_{(2)} = du^2 + \sin^2 u dv^2 \tag{1.75}$$

Thus we have two different Rimeannian manifolds $(D, G_{(1)})$ and $(D, G_{(2)})$. Metrics in (1.75) look similar. But.... It is easy to see that the first one is locally isometric to Euclidean plane, i.e. it is locally Euclidean Riemannian manifold since $\sin^2 v dv^2 = d(-\cos v)^2$: in new coordinates $u' = u, v' = \cos v$ Riemannian metric $G_{(1)}$ has appearance of standard Euclidean metric:

$$(du')^2 + (dv')^2 = (du)^2 + (d(\cos v))^2 = du^2 + \sin^2 v dv^2 = G_{(1)}\,.$$

This is not the case for second metric $G_{(2)}$. If we change notations $u \mapsto \theta$, $v \mapsto \varphi$ then $G_{(2)} = d\theta^2 + sin\theta^2 d\varphi^2$. This is local expression for Riemannian metric induced on the sphere of radius $R = 1$. Suppose that there exist coordinates $u' = u'(\theta, \varphi)$ $v' = v'(\theta, \varphi)$ such that in these coordinates metric has Eucldean appearance. This means that locally geometry of sphere is as a geometry of Euclidean plane. On the other hand we know from the course of

Geometry that this is not the case: sum of angles of triangels on the sphere is not equal to π, sphere cannot be bended without shrinking. Later in this course we will return to this question....

There are plenty other examples:

2) Plane with metric $\frac{4R^4(dx^2+dy^2)}{(R^2+x^2+y^2)^2}$ is isometric to the sphere with radius R.

3) Disc with metric $\frac{du^2+dv^2}{(1-u^2-v^2)^2}$ is isometric to half plane with metric $\frac{dx^2+dy^2}{4y^2}$.

(see also exercises in Homeworks and Coursework.)

1.5 Volume element in Riemannian manifold

The volume element in n-dimensional Riemannian manifold with metric $G = g_{ik}dx^i dx^k$ is defined by the formula

$$\sqrt{\det g}\, dx^1 dx^2 \dots dx^n \,. \tag{1.76}$$

If D is a domain in the n-dimensional Riemannian manifold with metric $G = g_{ik}dx^i$ then its volume is equal to to the integral of volume element over this domain.

$$V(D) = \int_D \sqrt{\det g}\, dx^1 dx^2 \dots dx^n \,. \tag{1.77}$$

Note that in the case of $n = 1$ volume is just the length, in the case if $n = 2$ it is area.

Why this formula for volume form? One can see that volume form (1.76) is invariant with respect to changing of coordinates i.e. if $y^1, \dots, y^n$ are new coordinates: $x^1 = x^1(y^1, \dots, y^n), x^2 = x^2(y^1, \dots, y^n)$...,

$$x^i = x^i(y^p), i = 1, \dots, n \quad , p = 1, \dots, n$$

and $\tilde{g}_{pq}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \,. \tag{1.78}$$

Then

$$\sqrt{\det g_{ik}(x)}\, dx^1 dx^2 \dots dx^n = \sqrt{\det \tilde{g}_{pq}(y)}\, dy^1 dy^2 \dots dy^n \tag{1.79}$$

This follows from (1.78). Namely

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)}\, dy^1 dy^2 \dots dy^n$$

Using the fact that $\det(ABC) = \det A \cdot \det B \cdot \det C$ and $\det\left(\frac{\partial x^i}{\partial y^p}\right) = \det\left(\frac{\partial x^k}{\partial y^q}\right)$[6] we see that from the formula above follows:

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det\left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q}\right)} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\left(\det\left(\frac{\partial x^i}{\partial y^p}\right)\right)^2} \sqrt{\det g_{ik}(x(y))} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\det g_{ik}(x(y))} \det\left(\frac{\partial x^i}{\partial y^p}\right) dy^1 dy^2 \dots dy^n = \tag{1.80}$$

Now note that

$$\det\left(\frac{\partial x^i}{\partial y^p}\right) dy^1 dy^2 \dots dy^n = dx^1 \dots dx^n$$

according to the formula for changing coordinates in n-dimensional integral [7]. Hence

$$\sqrt{\det g_{ik}(x(y))} \det\left(\frac{\partial x^i}{\partial y^p}\right) dy^1 dy^2 \dots dy^n = \sqrt{\det g_{ik}(x(y))} dx^1 dx^2 \dots dx^n \tag{1.81}$$

Thus we come to (1.79).

Remark Students who know the concept of exterior forms can read the volume element as n-form $\sqrt{\det g}\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$

In the next paragraph we will give another motivation of this formula from linear algebra.

1.5.1 Motivation: Gram formula for volume of parallelepiped

In this short paragraph we consider formulae for volume of n-dimensional parallelepiped, and we explain how formulae (1.76), (1.77) are related with basic formulae in geometry. For simplicity one can consider just the case if $n = 2, 3$.

Let $\mathbf{E}^n$ be Euclidean vector space equipped with orthonormal basis $\{\mathbf{e}_i\}$.

Let $\{\mathbf{a}_i\} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be an arbitrary row of n-vectors in $\mathbf{E}^n$. Consider n-dimensional parallelepiped $\Pi_{\{\mathbf{a}_i\}}$ formed by these vectors: $\Pi_{\{\mathbf{a}_i\}} : \mathbf{r} = t^i \mathbf{v}_i, 0 \le t^i \le 1$. The volume of this parallelepiped is equal to

$$Vol(\Pi_{\{\mathbf{a}_i\}}) = \det A\,, \tag{1.82}$$

[6] determinant of matrix does not change if we change the matrix on the adjoint, i.e. change columns on rows.

[7] Determinant of the matrix $\left(\frac{\partial x^i}{\partial y^p}\right)$ of changing of coordinates is called sometimes Jacobian. Here we consider the case if Jacobian is positive. If Jacobian is negative then formulae above remain valid just the symbol of modulus appears.

where the matrix $A = ||\mathbf{a}_i^m||$ is defined by expansion of vectors $\{\mathbf{a}_i\}$ over orthonormal basis $\{\mathbf{e}_i\}$: $\mathbf{a}_i = \mathbf{e}_m a_m^i$ (Volume vanishes ($Vol(\Pi_{\mathbf{a}_i}) = \det A = 0$) $\Leftrightarrow$ if $\{\mathbf{a}_i\}$ is not a basis.)

Now consider the scalar product (Riemannian metric) in $\mathbf{E}^n$ in the basis $\mathbf{a}_1, \dots, \mathbf{a}_n$:

$$g_{ik} = \langle \mathbf{a}_i, \mathbf{a}_k \rangle\,, \tag{1.83}$$

where $\langle\ ,\ \rangle$ is scalar product in $\mathbf{E}^n$: $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. We see that in (1.83)

$$g_{ij} = \langle \mathbf{a}_i, \mathbf{a}_k \rangle = \langle \sum_m a_i^m \mathbf{e}_m, \sum_n a_j^n \mathbf{e}_n \rangle = a_i^m \delta_{mn} a_i^n = (A^T \cdot A)_{ij} \Rightarrow \det G = (\det A)^2\,,$$

where $G = ||g_{ij}||$. Comparing with formula (1.82) we come to formula:

$$Vol(\Pi_{\mathbf{a}_i}) = \sqrt{\det g_{ik}} \tag{1.84}$$

This formula is called Gram formula, and the matrix $G = ||g_{ik}||$ is called Gram matrix for the vectors $\{\mathbf{a}_i\}$. Gram formula justifies equations (1.76) and (1.77) [8].

Remark One can easy see that formula (1.84) works for arbitrary n-dimensional parallelogramm in m-dimensional space. Indeed if $\alpha_1, \dots, \alpha_n$ are just arbitrary n vectors in m-dimensional Euclidean space then if $n < m$, the formula (1.82) s failed (matrix A is $m \times n$ matrix), but formula (1.84) works. For example the area of parallelogram formed by arbitrary vectors $\mathbf{a}_1, \mathbf{a}_2$ in $\mathbf{E}^n$ is equal to

$$\sqrt{\det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}} = \sqrt{\det \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \langle \mathbf{a}_1, \mathbf{a}_1 \rangle \\ \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \langle \mathbf{a}_1, \mathbf{a}_1 \rangle \end{pmatrix}}\,.$$

1.5.2 Examples of calculating volume element

Consider first very simple example: Volume element of plane in Cartesian coordinates, metric $g = dx^2 + dy^2$. Volume element is equal to

$$\sqrt{\det g}dxdy = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} dxdy = dxdy$$

[8] We see that n-dimensional parallelepiped $\Pi_{\{\mathbf{a}_i\}}$ in new coordinates t^i corresponding to the basis $\{\mathbf{a}_i\}$ becomes n-dimensional cube, Standard Euclidean metric $G = dx^i \delta_{ik} dx^k$ (in orthonormal basis $\{e_i\}$) transforms to $G = dx^i \delta_{ik} dx^k = (a_m^i dt^m) \delta_{ik} (a_n^k dt^n) = (A^T A)_{mn} dt^m dt^n$ and

$$\text{Volume}\Pi_{\{\mathbf{a}_i\}} = \int_{\mathbf{x} \in \Pi} dx^1 \dots dx^n = \int_{0 \le t_i \le 1} \sqrt{G} dt^1 \dots dt^n = \sqrt{\det G} = \sqrt{\det A^T A}\,.$$

Volume of the domain D is equal to

$$V(D) = \int_D \sqrt{\det g} dxdy = \int_D dxdy$$

If we go to polar coordinates:

$$x = r\cos\varphi, y = r\sin\varphi \tag{1.85}$$

Then we have for metric:

$$G = dr^2 + r^2 d\varphi^2$$

because

$$dx^2 + dy^2 = (dr\cos\varphi - r\sin\varphi d\varphi)^2 + (dr\sin\varphi + r\cos\varphi d\varphi)^2 = dr^2 + r^2 d\varphi^2 \tag{1.86}$$

Volume element in polar coordinates is equal to

$$\sqrt{\det g} drd\varphi = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}} drd\varphi = rdrd\varphi\,.$$

Lobachesvky plane.

In coordinates x, y $(y > 0)$ metric $G = \frac{dx^2+dy^2}{y^2}$, the corresponding matrix $G = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$. Volume element is equal to $\sqrt{\det g} dxdy = \frac{dxdy}{y^2}$.

Sphere in stereographic coordinates In stereographic coordinates

$$G = \frac{4R^4(du^2 + dv^2)}{(R + u^2 + v^2)^2} \tag{1.87}$$

(It is isometric to the sphere of the radius R without North pole in stereographic coordinates (see the Homeworks.))

Calculate its volume element and volume. It is easy to see that:

$$G = \begin{pmatrix} \frac{4R^4}{(R^2+u^2+v^2)^2} & 0 \\ 0 & \frac{4R^4}{(R^2+u^2+v^2)^2} \end{pmatrix} \qquad \det g = \frac{16R^8}{(R^2 + u^2 + v^2)^4} \tag{1.88}$$

and volume element is equal to $\sqrt{\det g} dudv = \frac{4R^4 dudv}{(R^2+u^2+v^2)^2}$

One can calculate volume in coordinates u, v but it is better to consider homothety $u \to Ru, v \to Rv$ and polar coordinates: $u = Rr\cos\varphi, v = Rr\sin\varphi$. Then volume form is equal to $\sqrt{\det g}dudv = \frac{4R^4 dudv}{(R^2+u^2+v^2)^2} = \frac{4R^2 rdrd\varphi}{(1+r^2)^2}$.
Now calculation of integral becomes easy:

$$V = \int \frac{4R^2 rdrd\varphi}{(1+r^2)^2} = 8\pi R^2 \int_0^\infty \frac{rdr}{(1+r^2)^2} = 4\pi R^2 \int_0^\infty \frac{du}{(1+u)^2} = 4\pi R^2\,.$$

Domain in Lobachevsky plane.
Consider in Lobachevsky plane the domain D_a such that

$$D_a = \{x, y\colon \quad x^2 + y^2 \geq 1, |y| \leq a\}\,, (|a| \leq 1)\,. \tag{1.89}$$

Remark Note that vertical lines and half-circle are geodesics. One can see that the distance between these lines tends to zero. (We will study it later). If we denote by A a point $(a, \sqrt{1-a^2})$ and by B the point $(-a, \sqrt{1-a^2})$, then the domain D_a can be cinsidered as a 'triangle' with vertices at the point A, B, C where C is a point at infinity. The meaning of this remark we will study later.

One can calculate the area of this domain, using area form on Lobachevsky plane

$$V(D_a) = \int_{-a\leq y\leq a\,, x^2+y^2\geq 1} \frac{dxdy}{y^2} = 2\arcsin a \tag{1.90}$$

(See in detail Homework) We will discuss later the geometrical meaning of this formula.

Segment of the sphere.

Consider sphere of the radius a in Euclidean space with standard Riemannian metric

$$a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2$$

This metric is nothing but first quadratic form on the sphere (see (1.3.3)). The volume element is

$$\sqrt{\det g}d\theta d\varphi = \sqrt{\det\begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin\theta \end{pmatrix}}d\theta d\varphi = a^2\sin\theta d\theta d\varphi$$

Now calculate the volume of the segment of the sphere between two parallel planes, i.e. domain restricted by parallels $\theta_1 \leq \theta \leq \theta_0$: Denote by h be the height of this segment. One can see that

$$h = a\cos\theta_0 - a\cos\theta_1 = a(\cos\theta_0 - a\cos\theta_1)$$

There is remarkable formula which express the area of segment via the height h:

$$V=\int_{\theta_1\leq\theta\leq\theta_0}\left(a^2\sin\theta\right)d\theta d\varphi=\int_{\theta^0}^{\theta^1}\left(\int_0^{2\pi}\left(a^2\sin\theta\right)d\varphi\right)d\theta=$$

$$\int_{\theta^1}^{\theta^0}2\pi a^2\sin\theta d\theta=2\pi a^2(\cos\theta_0-\cos\theta_1)=2\pi a(a\cos\theta_0-acos\theta_1)=2\pi ah \tag{1.91}$$

E.g. for all the sphere $h=2a$. We come to $S=4\pi a^2$. It is remarkable formula: area of the segment is a polynomial function of radius of the sphere and height (Compare with formula for length of the arc of the circle)

2 Covariant differentiaion. Connection. Levi Civita Connection on Riemannian manifold

2.1 Differentiation of vector field along the vector field.— Affine connection

How to differentiate vector fields on a (smooth)manifold M?

Recall the differentiation of functions on a (smooth)manifold M.

Let $\mathbf{X}=\mathbf{X^i}(\mathbf{x})\mathbf{e_i}(\mathbf{x})=\frac{\partial}{\partial\mathbf{x^i}}$ be a vector field on M. Recall that vector field [9] $\mathbf{X}=\mathbf{X^i}\mathbf{e_i}$ defines at the every point x_0 an infinitesimal curve: $x^i(t)=x_0^i+tX^i$ (More exactly the equivalence class $[\gamma(t)]_{\mathbf{X}}$ of curves $x^i(t)=x_0^i+tX^i+\dots$).

Let f be an arbitrary (smooth) function on M and $\mathbf{X}=X^i\frac{\partial}{\partial x^i}$. Then derivative of function f along vector field $\mathbf{X}=X^i\frac{\partial}{\partial x^i}$ is equal to

$$\partial_{\mathbf{X}}f=\nabla_{\mathbf{X}}f=X^i\frac{\partial f}{\partial x^i}$$

The geometrical meaning of this definition is following: If $\mathbf{X}$ is a velocity vector of the curve $x^i(t)$ at the point $x_0^i=x^i(t)$ at the "time" $t=0$ then the value of the derivative $\nabla_{\mathbf{X}}f$ at the point $x_0^i=x^i(0)$ is equal just to the

[9] here like always we suppose by default the summation over repeated indices. E.g.$\mathbf{X}=X^i\mathbf{e}_i$ is nothing but $\mathbf{X}=\sum_{i=1}^n X^i\mathbf{e}_i$

derivative by t of the function $f(x^i(t))$ at the "time" $t=0$:

$$\text{if} \quad X^i(x)\big|_{x_0=x(0)} = \frac{dx^i(t)}{dt}\big|_{t=0}, \quad \text{then} \quad \nabla_{\mathbf{X}} f\big|_{x^i=x^i(0)} = \frac{d}{dt} f\left(x^i\left(t\right)\right)\big|_{t=0} \tag{2.1}$$

Remark In the course of Geometry and Differentiable Manifolds the operator of taking derivation of function along the vector field was denoted by "$\partial_{\mathbf{X}} f$". In this course we prefer to denote it by "$\nabla_{\mathbf{X}} f$" to have the uniform notation for both operators of taking derivation of functions and vector fields along the vector field.

One can see that the operation ∇_X on the space $C^\infty(M)$ (space of smooth functions on the manifold) satisfies the following conditions:

- $\nabla_{\mathbf{X}}\left(\lambda f + \mu g\right) = \lambda \nabla_{\mathbf{X}} f + \mu \nabla_{\mathbf{X}} g$ where $\lambda, \mu \in \mathbf{R}$ (linearity over numbers)
- $\nabla_{h\mathbf{X}+g\mathbf{Y}}(f) = h\nabla_{\mathbf{X}}(f) + g\nabla_{\mathbf{Y}}(f)$ (linearity over the space of functions)
- $\nabla_{\mathbf{X}}(\lambda f g) = f\nabla_{\mathbf{X}}(\lambda g) + g\nabla_{\mathbf{X}}(\lambda f)$ (Leibnitz rule)

(2.2)

Remark One can prove that these properties characterize vector fields:operator on smooth functions obeying the conditions above is a vector field. (You will have a detailed analysis of this statement in the course of Differentiable Manifolds.)

How to define differentiation of vector fields along vector fields.

The formula (2.1) cannot be generalised straightforwardly because vectors at the point x_0 and $x_0 + tX$ are vectors from different vector spaces. (We cannot substract the vector from one vector space from the vector from the another vector space, because *apriori* we cannot compare vectors from different vector space. One have to define an operation of transport of vectors from the space $T_{x_0}M$ to the point $T_{x_0+tX}M$ defining the transport from the point $T_{x_0}M$ to the point $T_{x_0+tX}M$).

Try to define the operation ∇ on vector fields such that conditions (2.2) above be satisfied.

2.1.1 Definition of connection. Christoffel symbols of connection

Definition Affine connection on M is the *operation* ∇ which assigns to every vector field $\mathbf{X}$ a linear map, (but not necessarily $C(M)$-linear map!) (i.e. a map which is linear over numbers not necessarily over functions) $\nabla_{\mathbf{X}}$ on the space of vector fields on M:

$$\nabla_{\mathbf{X}} (\lambda \mathbf{Y} + \mu \mathbf{Z}) = \lambda \nabla_{\mathbf{X}} \mathbf{Y} + \mu \nabla_{\mathbf{X}} \mathbf{Z}, \qquad \text{for every } \lambda, \mu \in \mathbf{R} \tag{2.3}$$

(Compare the first condition in (2.2)).
which satisfies the following conditions:

- for arbitrary (smooth) functions f, g on M

  $$\nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}} (\mathbf{Z}) = f \nabla_{\mathbf{X}} (\mathbf{Z}) + g \nabla_{\mathbf{Y}} (\mathbf{Z}) \qquad (C^\infty(M)\text{-linearity}) \tag{2.4}$$

  (compare with second condition in (2.2))

- for arbitrary function f

  $$\nabla_{\mathbf{X}} (f\mathbf{Y}) = (\nabla_{\mathbf{X}} f)\, \mathbf{Y} + f \nabla_{\mathbf{X}} (\mathbf{Y}) \qquad (\text{Leibnitz rule}) \tag{2.5}$$

  Recall that $\nabla_{\mathbf{X}} f$ is just usual derivative of a function f along vector field: $\nabla_{\mathbf{X}} f = \partial_{\mathbf{X}} f$.

  (Compare with Leibnitz rule in (2.2)).

  The operation $\nabla_{\mathbf{X}}\mathbf{Y}$ is called covariant derivative of vector field $\mathbf{Y}$ along the vector field $\mathbf{X}$.

Write down explicit formulae in a given local coordinates $\{x^i\}$ ($i = 1, 2, \ldots, n$) on manifold M.

Let

$$\mathbf{X} = X^i \mathbf{e}_i = X^i \frac{\partial}{\partial x^i} \qquad \mathbf{Y} = Y^i \mathbf{e}_i = Y^i \frac{\partial}{\partial x^i}$$

The basis vector fields $\frac{\partial}{\partial x^i}$ we denote sometimes by ∂_i sometimes by $\mathbf{e}_i$

Using properties above one can see that

$$\nabla_{\mathbf{X}} \mathbf{Y} = \nabla_{X^i \partial_i} Y^k \partial_k = X^i \left(\nabla_i \left(Y^k \partial_k \right) \right), \qquad \text{where } \nabla_i = \nabla_{\partial_i} \tag{2.6}$$

Then according to (2.4)

$$\nabla_i \left(Y^k \partial_k \right) = \nabla_i \left(Y^k \right) \partial_k + Y^k \nabla_i \partial_k$$

Decompose the vector field $\nabla_i \partial_k$ over the basis ∂_i:

$$\nabla_i \partial_k = \Gamma^m_{ik} \partial_m \tag{2.7}$$

and

$$\nabla_i \left(Y^k \partial_k\right) = \frac{\partial Y^k(x)}{\partial x^i} \partial_k + Y^k \Gamma^m_{ik} \partial_m, \tag{2.8}$$

$$\nabla_{\mathbf{X}} \mathbf{Y} = X^i \frac{\partial Y^m(x)}{\partial x^i} \partial_m + X^i Y^k \Gamma^m_{ik} \partial_m, \tag{2.9}$$

In components

$$(\nabla_{\mathbf{X}} \mathbf{Y})^m = X^i \left(\frac{\partial Y^m(x)}{\partial x^i} + Y^k \Gamma^m_{ik} \right) \tag{2.10}$$

Coefficients $\{\Gamma^m_{ik}\}$ are called *Christoffel symbols* in coordinates $\{x^i\}$. These coefficients define covariant derivative—**connection**.

If operation of taking covariant derivative is given we say that the connection is given on the manifold. Later it will be explained why we us the word "connection"

We see from the formula above that to define covariant derivative of vector fields, connection, we have to define Christoffel symbols in local coordinates.

2.1.2 Transformation of Christoffel symbols for an arbitrary connection

Let ∇ be a connection on manifold M. Let $\{\Gamma^i_{km}\}$ be Christoffel symbols of this connection in given local coordinates $\{x^i\}$. Then according (2.7) and (2.8) we have

$$\nabla_{\mathbf{X}} \mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m} \frac{\partial}{\partial x^i} + X^m \Gamma^i_{mk} Y^k \frac{\partial}{\partial x^i},$$

and in particularly

$$\Gamma^i_{mk} \partial_i = \nabla_{\partial_m} \partial_k$$

Use this relation to calculate Christoffel symbols in new coordinates $x^{i'}$

$$\Gamma^{i'}_{m'k'} \partial_{i'} = \nabla_{\partial'_m} \partial_{k'}$$

We have that $\partial_{m'} = \frac{\partial}{\partial x^{m'}} = \frac{\partial x^m}{\partial x^{m'}} \frac{\partial}{\partial x^m} = \frac{\partial x^m}{\partial x^{m'}} \partial_m$. Hence due to properties (2.4), (2.5) we have

$$\Gamma^{i'}_{m'k'} \partial_{i'} = \nabla_{\partial_{m'}} \partial_{k'} = \nabla_{\partial'_m} \left(\frac{\partial x^k}{\partial x^{k'}} \partial_k \right) = \left(\frac{\partial x^k}{\partial x^{k'}} \right) \nabla_{\partial'_m} \partial_k + \frac{\partial}{\partial x^{m'}} \left(\frac{\partial x^k}{\partial x^{k'}} \right) \partial_k =$$

$$\left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\frac{\partial x^m}{\partial x^{m'}}\partial_m}\partial_k+\frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k=\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\nabla_{\partial_m}\partial_k+\frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k$$

$$\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\partial_i+\frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k=\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\frac{\partial x^{i'}}{\partial x^i}\partial_{i'}+\frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^k}\partial_{i'}$$

Comparing the first and the last term in this formula we come to the transformation law:

If $\{\Gamma^i_{km}\}$ are Christoffel symbols of the connection ∇ in local coordinates $\{x^i\}$ and $\{\Gamma^{i'}_{k'm'}\}$ are Christoffel symbols of this connection in new local coordinates $\{x^{i'}\}$ then

$$\Gamma^{i'}_{k'm'}=\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{km}+\frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} \tag{2.11}$$

Remark Christoffel symbols do not transform as tensor. If the second term is equal to zero, i.e. transformation of coordinates are linear (see the Proposition on flat connections) then the transformation rule above is the the same as a transformation rule for tensors of the type $\begin{pmatrix}1\\2\end{pmatrix}$ (see the formula (1.11)). In general case this is not true. Christoffel symbols do not transform as tensor under arbitrary non-linear coordinate transformation: see the second term in the formula above.

Remark On the other hand note that *difference of two arbitrary connections is a tensor.* If Γ^i_{km} and $\tilde{\Gamma}^i_{km}$ are corresponding Chrstoffel symbols then it follows from (1.11)) that their difference $T^i_{km}=\Gamma^i_{km}-\tilde{\Gamma}^i_{km}$ transforms as a tensor:

$$T^{i'}_{k'm'}=\Gamma^{i'}_{k'm'}-\tilde{\Gamma}^{i'}_{k'm'}=\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\left(\Gamma^i_{km}-\tilde{\Gamma}^i_{km}\right)=\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}T^i_{km}$$

(See for detail the Homework 5.)

2.1.3 Canonical flat affine connection

It follows from the properties of connection that it is suffice to define connection at vector fields which form basis at the every point using (2.7), i.e. to define Christoffel symbols of this connection.

Example Consider n-dimensional Euclidean space $\mathbf{E}^n$ with Cartesian coordinates $\{x^1,\dots,x^n\}$.

Define connection such that all Christoffel symbols are equal to zero in these Cartesian coordinates $\{x^i\}$.

$$\nabla_{\mathbf{e}_i}\mathbf{e}_k=\Gamma^m_{ik}\mathbf{e}_m=0,\quad \Gamma^m_{ik}=0 \tag{2.12}$$

Does this mean that Christoffel symbols are equal to zero in an arbitrary Cartesian coordinates if they equal to zero in given Cartesian coordinates?

Does this mean that Christoffel symbols of this connection equal to zero in arbitrary coordinates system?

it follows from transformation rules (2.11) for Christoffel symbols that Christofel symbols vanish also in new coordinates $x^{i'}$ if and only if

$$\frac{\partial^2 x^i}{\partial x^{m'} \partial x^{i'}} = 0, \text{ i.e. } x^i = b^i + a^i_k x^k \tag{2.13}$$

i.e. the relations between new and old coordinates are linear. We come to simple but very important

Proposition *Let all Christoffel symbols of a given connection be equal to zero in a given coordinate system $\{x^i\}$. Then all Christoffel symbols of this connection are equal to zero in an arbitrary coordinate system $\{x^{i'}\}$ such that the relations between new and old coordinates are linear:*

$$x^{i'} = b^i + a^i_k x^k \tag{2.14}$$

If transformation to new coordinate system is not linear, i.e. $\frac{\partial^2 x^i}{\partial x^{m'} \partial x^{i'}} \neq 0$ then Christoffel symbols of this connection in general are not equal to zero in new coordinate system $\{x^{i'}\}$.

Definition We call connection ∇ flat if there exists coordinate system such that all Christoffel symbols of this connection are equal to zero in a given coordinate system.

In particular connection (2.12) has zero Christoffel symbols in arbitrary Cartesian coordinates.

Corollary Connection has zero Christoffel symbols in arbitrary Cartesian coordinates if it has zero Christoffel symbols in a given Cartesian coordinates.

Hence the following definition is correct:

Definition A connection on the Euclidean space $\mathbf{E}^n$ which Christoffel symbols vanish in Cartesian coordinates is called *canonical flat connection.*

Remark Canonical flat connection in Euclidean space is uniquely defined, since Cartesian coordinates are defined globally. On the other hand on arbitrary manifold one can define flat connection locally just choosing any arbitrary *local* coordinates and define *locally flat connection* by condition that Christoffel symbols vanish in these local coordinates. This does not mean that one can define flat connection *globally.* We will study this question after learning transformation law for Christoffel symbols.

Remark One can see that flat connection is symmetric connection.

Example Consider a connection (2.12) in $\mathbf{E}^2$. It is a flat connection. Calculate Christoffel symbols of this connection in polar coordinates

$$\begin{cases} x = r\cos\varphi \\ y = y\sin\varphi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases} \tag{2.15}$$

Write down Jacobians of transformations—matrices of partial derivatives:

$$\begin{pmatrix} x_r & y_r \\ x_\varphi & y_\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix}, \qquad \begin{pmatrix} r_x & \varphi_x \\ r_y & \varphi_y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} \tag{2.16}$$

According (2.11) and since Chrsitoffel symbols are equal to zero in Cartesian coordinates (x, y) we have

$$\Gamma^{i'}_{k'm'} = \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}, \tag{2.17}$$

where $(x^1, x^2) = (x, y)$ and $(x^{1'}, x^{2'}) = (r, \varphi)$. Now using (2.16) we have

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial r\partial r}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial r}\frac{\partial r}{\partial y} = 0$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial r}{\partial y} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0\,.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial\varphi\partial\varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial r}{\partial y} = -x\frac{x}{r} - y\frac{y}{r} = -r\,.$$

$$\Gamma^\varphi_{rr} = \frac{\partial^2 x}{\partial r\partial r}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial r\partial r}\frac{\partial\varphi}{\partial y} = 0\,.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial\varphi}{\partial y} = -\sin\varphi\frac{-y}{r^2} + \cos\varphi\frac{x}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial\varphi\partial\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial\varphi\partial\varphi}\frac{\partial\varphi}{\partial y} = -x\frac{-y}{r^2} - y\frac{x}{r^2} = 0\,. \tag{2.18}$$

Hence we have that the covariant derivative (2.12) in polar coordinates has the following appearance

$$\nabla_r \partial_r = \Gamma^r_{rr}\partial_r + \Gamma^\varphi_{rr}\partial_\varphi = 0\,,, \qquad \nabla_r \partial_\varphi = \Gamma^r_{r\varphi}\partial_r + \Gamma^\varphi_{r\varphi}\partial_\varphi = \frac{\partial_\varphi}{r}$$

$$\nabla_\varphi \partial_r = \Gamma^r_{\varphi r}\partial_r + \Gamma^\varphi_{\varphi r}\partial_\varphi = \frac{\partial_\varphi}{r}, \qquad \nabla_\varphi \partial_\varphi = \Gamma^r_{\varphi\varphi}\partial_r + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = -r\partial_r \quad (2.19)$$

Remark Later when we study geodesics we will learn a very quick method to calculate Christoffel symbols.

2.2 Connection induced on the surfaces

Let M be a manifold embedded in Euclidean spaceCanonical flat connection on $\mathbf{E}^N$ induces the connection on surface in the following way.

Let $\mathbf{X}, \mathbf{Y}$ be tangent vector fields to the surface M and $\nabla^{\text{can.flat}}$ a canonical flat connection in $\mathbf{E}^N$. In general

$$\mathbf{Z} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y} \quad \text{is not tangent to manifold } M \qquad (2.20)$$

Consider its decomposition on two vector fields:

$$\mathbf{Z} = \mathbf{Z}_{tangent} + \mathbf{Z}_\perp, \nabla^{\text{can.flat}}_{\mathbf{X}}, \mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} + \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_\perp\,, \quad (2.21)$$

where $\mathbf{Z}_\perp$ is a component of vector which is orthogonal to the surface M and $\mathbf{Z}_{||}$ is a component which is tangent to the surface. Define an induced connection ∇^M on the surface M by the following formula

$$\nabla^M: \qquad \nabla^M_{\mathbf{X}}\mathbf{Y}: \,= \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} \qquad (2.22)$$

One can see that this formula really defines the connection on surface M, i.e. the operation defined by this relation obeys all axioms of connection. Indeed it is easy to see that for arbitrary vector fields $\mathbf{X}$ and $\mathbf{Y}$, the vector field $\nabla^M_{\mathbf{X}}\mathbf{Y}$ is tangent vector field, and this operation obeys relations (2.3), (2.4) and (2.5). For example check Leibnitz rule:

$$\nabla^M_{\mathbf{X}}(f\mathbf{Y}) = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}(f\mathbf{Y})\right)_{tangent} = \left(\partial_{\mathbf{X}} f\mathbf{Y} + f\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} =$$

$$\partial_{\mathbf{X}} f\mathbf{Y} + f\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}_{tangent} = \partial_{\mathbf{X}} f\mathbf{Y} + f\nabla^M_{\mathbf{X}}\mathbf{Y}\,.$$

We mainly apply this construction for 2-dimensional manifolds (surfaces) in $\mathbf{E}^3$.

2.2.1 Calculation of induced connection on surfaces in $\mathbf{E}^3$.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$. Let $\nabla^{\text{can.flat}}$ be a flat connection in $\mathbf{E}^3$. Then

$$\nabla^M\colon \qquad \nabla^M_{\mathbf{X}}\mathbf{Y}\colon = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{||} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y} - \mathbf{n}(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}, \mathbf{n}), \tag{2.23}$$

where $\mathbf{n}$ is normal unit vector field to M. Consider a special example

Example (Induced connection on sphere) Consider a sphere of the radius R in $\mathbf{E}^3$:

$$\mathbf{r}(\theta, \varphi)\colon \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$$

then

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} sin\theta\cos\varphi \\ sin\theta\sin\varphi \\ \cos\theta \end{pmatrix},$$

where $\mathbf{r}_\theta = \frac{\partial\mathbf{r}(\theta,\varphi)}{\partial\theta}$, $\mathbf{r}_\varphi = \frac{\partial\mathbf{r}(\theta,\varphi)}{\partial\varphi}$ are basic tangent vectors and $\mathbf{n}$ is normal unit vector.

Calculate an induced connection ∇ on the sphere.

First calculate $\nabla_{\partial_\theta}\partial_\theta$.

$$\nabla_{\partial_\theta}\partial_\theta = \left(\frac{\partial\mathbf{r}_\theta}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\theta})_{tangent}\,.$$

On the other hand one can see that $\mathbf{r}_{\theta\theta} = \begin{pmatrix} -Rsin\theta\cos\varphi \\ -Rsin\theta\sin\varphi \\ -R\cos\theta \end{pmatrix} = -R\mathbf{n}$ is proportional to normal vector, i.e. $(\mathbf{r}_{\theta\theta})_{tangent} = 0$. We come to

$$\nabla_{\partial_\theta}\partial_\theta = (\mathbf{r}_{\theta\theta})_{tangent} = 0 \Rightarrow \Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0\,. \tag{2.24}$$

Remark Notice that equation (2.24) follows from teh fact that $\mathbf{r}_{\theta\theta}$ is centripetal acceleration which is directed along $\mathbf{r} \sim \mathbf{n}$.

Now calculate $\nabla_{\partial_\theta}\partial_\varphi$ and $\nabla_{\partial_\varphi}\partial_\theta$.

$$\nabla_{\partial_\theta}\partial_\varphi = \left(\frac{\partial\mathbf{r}_\varphi}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\varphi})_{tangent}\,, \nabla_{\partial_\varphi}\partial_\theta = \left(\frac{\partial\mathbf{r}_\theta}{\partial\varphi}\right)_{tangent} = (\mathbf{r}_{\varphi\theta})_{tangent}$$

We have

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix}_{tangent} .$$

We see that the vector $\mathbf{r}_{\varphi\theta}$ is orthogonal to $\mathbf{n}$:

$$\langle \mathbf{r}_{\varphi\theta}, \mathbf{n}\rangle = -R\cos\theta\sin\varphi\sin\theta\cos\varphi + R\cos\theta\cos\varphi\sin\theta\sin\varphi = 0.$$

Hence

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \mathbf{r}_{\varphi\theta} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix} = \operatorname{cotan}\theta\mathbf{r}_\varphi .$$

We come to

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \operatorname{cotan}\theta\partial_\varphi \Rightarrow \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0,\ \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \operatorname{cotan}\theta \quad (2.25)$$

Finally calculate $\nabla_{\partial_\varphi}\partial_\varphi$

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \left(\begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix}\right)_{tangent}$$

Projecting on the tangent vectors to the sphere (see (2.23)) we have

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle \mathbf{n}, \mathbf{r}_{\varphi\varphi}\rangle =$$

$$\begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix} - \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} (-R\sin\theta\cos\varphi\sin\theta\cos\varphi - R\sin\theta\sin\varphi\sin\theta\sin\varphi) =$$

$$-\sin\theta\cos\theta \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix} = -\sin\theta\cos\theta\mathbf{r}_\theta,$$

i.e.

$$\nabla_{\partial_\varphi}\partial_\varphi = -\sin\theta\cos\theta\mathbf{r}_\theta \Rightarrow \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta,\ \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{\varphi\varphi} = 0 . \qquad (2.26)$$

2.3 Levi-Civita connection

2.3.1 Symmetric connection

Definition. We say that connection is symmetric if its Christoffel symbols Γ^i_{km} are symmetric with respect to lower indices

$$\Gamma^i_{km} = \Gamma^i_{mk} \tag{2.27}$$

The canonical flat connection and induced connections considered above are symmetric connections.

Invariant definition of symmetric connection

A connection ∇ is symmetric if for an arbitrary vector fields $\mathbf{X}, \mathbf{Y}$

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0 \tag{2.28}$$

If we apply this definition to basic fields ∂_k, ∂_m which commute: $[\partial_k, \partial_m] = 0$ we come to the condition

$$\nabla_{\partial_k}\partial_m - \nabla_{\partial_m}\partial_k = \Gamma^i_{mk}\partial_i - \Gamma^i_{km}\partial_i = 0$$

and this is the condition (2.27).

2.3.2 Levi-Civita connection. Theorem and Explicit formulae

Let (M, G) be a Riemannian manifold.

Definition-Theorem

A symmetric connection ∇ is called Levi-Civita connection if it is compatible with metric, i.e. if it preserves the scalar product:

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle = \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}\rangle \tag{2.29}$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

There exists unique levi-Civita connection on the Riemannian manifold.

In local coordinates Christoffel symbols of Levi-Civita connection are given by the following formulae:

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right) . \tag{2.30}$$

where $G = g_{ik}dx^i dx^k$ is Riemannian metric in local coordinates and $||g^{ik}||$ is the matrix inverse to the matrix $||g_{ik}||$.

Proof

Suppose that this connection exists and Γ^i_{mk} are its Christoffel symbols. Consider vector fields $\mathbf{X} = \partial_m, \mathbf{Y} = \partial_i$ and $\mathbf{Z} = \partial_k$ in (2.29). We have that

$$\partial_m g_{ik} = \langle \Gamma^r_{mi}\partial_r, \partial_k\rangle + \langle \partial_i, \Gamma^r_{mk}\partial_r\rangle = \Gamma^r_{mi} g_{rk} + g_{ir}\Gamma^r_{mk}\,. \tag{2.31}$$

for arbitrary indices m, i, k.

Denote by $\Gamma_{mik} = \Gamma^r_{mi} g_{rk}$ we come to

$$\partial_m g_{ik} = \Gamma_{mik} + \Gamma_{mki},\ \text{i.e.}$$

Now using the symmetricity $\Gamma_{mik} = \Gamma_{imk}$ since $\Gamma^k_{mi} = \Gamma^k_{im}$ we have

$$\Gamma_{mik} = \partial_m g_{ik} - \Gamma_{mki} = \partial_m g_{ik} - \Gamma_{kmi} = \partial_m g_{ik} - (\partial_k g_{mi} - \Gamma_{kim}) =$$

$$\partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{kim} = \partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{ikm} = \partial_m g_{ik} - \partial_k g_{mi} + (\partial_i g_{km} - \Gamma_{imk}) =$$

$$\partial_m g_{ik} - \partial_k g_{mi} + \partial_i g_{km} - \Gamma_{mik}\,.$$

Hence

$$\Gamma_{mik} = \frac{1}{2}(\partial_m g_{ik} + \partial_i g_{mk} - \partial_k g_{mi}) \Rightarrow \Gamma^k_{im} = \frac{1}{2} g^{kr}\left(\partial_m g_{ir} + \partial_i g_{mr} - \partial_r g_{mi}\right) \tag{2.32}$$

We see that if this connection exists then it is given by the formula(2.30).

On the other hand one can see that (2.30) obeys the condition (2.31). We prove the uniqueness and existence.

since $\nabla_{\partial_i}\partial_k = \Gamma^m_{ik}\partial_m$.

Consider examples.

2.3.3 Levi-Civita connection of $\mathbf{E}^n$

For Euclidean space $\mathbf{E}^n$ in standard Cartesian coordinates

$$G_{\text{Eucl}} = (dx^1)^2 + \cdots + (dx^n)^2 = \delta_{ik} dx^i dx^k$$

Components of metric are constants (they are equal to 0 or 1). Hence obviously Christoffel symbols of Levi-Civita connection in Cartesian coordinates according formula (2.30) vanish:

$$\Gamma^I_{km} = 0 \text{ in Cartesian coordiantes}$$

Recalling canonical flat connection (see 2.1.3) we come to simple but important observation:

Observation Levi-Civita connection coincides with canonical flat connection in Euclidean space $\mathbf{E}^n$. They have vanishing Cristoffel symbols in Cartesian coordinates.

2.3.4 Levi-Civita connection on 2-dimensional Riemannian manifold with metric $G = adu^2 + bdv^2$.

Example Consider 2-dimensional manifold with Riemannian metrics

$$G = a(u,v)du^2 + b(u,v)dv^2, \qquad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a(u,v) & 0 \\ 0 & b(u,v) \end{pmatrix}$$

Calculate Christoffel symbols of Levi Civita connection.

Using (2.32) we see that:

$$\begin{aligned}
\Gamma_{111} &= \tfrac{1}{2}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \tfrac{1}{2}\partial_1 g_{11} = & \tfrac{1}{2}a_u \\
\Gamma_{211} = \Gamma_{121} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}\right) = \tfrac{1}{2}\partial_2 g_{11} = & \tfrac{1}{2}a_v \\
\Gamma_{221} &= \tfrac{1}{2}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) = -\tfrac{1}{2}\partial_1 g_{22} = - & \tfrac{1}{2}b_u \\
\Gamma_{112} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right) = -\tfrac{1}{2}\partial_2 g_{11} = - & \tfrac{1}{2}a_v \\
\Gamma_{122} = \Gamma_{212} &= \tfrac{1}{2}\left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21}\right) = \tfrac{1}{2}\partial_1 g_{22} = & \tfrac{1}{2}b_u \\
\Gamma_{222} &= \tfrac{1}{2}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) = \tfrac{1}{2}\partial_2 g_{22} = & \tfrac{1}{2}b_v
\end{aligned} \tag{2.33}$$

To calculate $\Gamma^i_{km} = g^{ir}\Gamma_{kmr}$ note that for the metric $a(u,v)du^2 + b(u,v)dv^2$

$$G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{a(u,v)} & 0 \\ 0 & \frac{1}{b(u,v)} \end{pmatrix}$$

Hence

$$\begin{aligned}
&\Gamma^1_{11} = g^{11}\Gamma_{111} = \tfrac{a_u}{2a}, \quad &&\Gamma^1_{21} = \Gamma^1_{12} = g^{11}\Gamma_{121} = \tfrac{a_v}{2a}, \quad &&\Gamma^1_{22} = g^{11}\Gamma_{221} = \tfrac{-b_u}{2a} \\
&\Gamma^2_{11} = g^{22}\Gamma_{112} = \tfrac{-a_v}{2b}, \quad &&\Gamma^2_{21} = \Gamma^2_{12} = g^{22}\Gamma_{122} = \tfrac{b_u}{2b}, \quad &&\Gamma^2_{22} = g^{22}\Gamma_{222} = \tfrac{b_v}{2b}
\end{aligned} \tag{2.34}$$

2.3.5 Example of the sphere again

Calculate Levi-Civita connection on the sphere.

On the sphere first quadratic form (Riemannian metric) $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Hence we use calculations from the previous example with

$a(\theta,\varphi) = R^2, b(\theta,\varphi) = R^2 \sin^2\theta$ $(u=\theta, v=\varphi)$. Note that $a_\theta = a_\varphi = b_\varphi = 0$. Hence only non-trivial components of Γ will be:

$$\Gamma^\theta_{\varphi\varphi} = \frac{-b_\theta}{2a} = \frac{-\sin 2\theta}{2}, \qquad \left(\Gamma_{\varphi\varphi\theta} = \frac{-R^2 \sin 2\theta}{2}\right), \tag{2.35}$$

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{b_\theta}{2b} = \frac{\cos\theta}{\sin\theta} \qquad \left(\Gamma_{\theta\varphi\varphi} = \frac{R^2 \sin 2\theta}{2}\right) \tag{2.36}$$

All other components are equal to zero:

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \Gamma^\varphi_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0$$

Remark Note that Christoffel symbols of Levi-Civita connection on the sphere coincide with Christoffel symbols of induced connection calculated in the subsection "Connection induced on surfaces". later we will understand the geometrical meaning of this fact.

2.4 Levi-Civita connection = induced connection on surfaces in $\mathbf{E}^3$

We know already that *canonical flat connection of Euclidean space is the Levi-Civita connection of the standard metric on Euclidean space.* (see section 2.3.3.) Now we show that Levi-Civita connection on surfaces in Euclidean space coincides with the connection induced on the surfaces by canonical flat connection. We perform our analysis for surfaces in $\mathbf{E}^3$.

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface in $\mathbf{E}^3$. Let G be induced Riemannian metric on M and ∇ Levi–Civita connection of this metric.

We know that the induced connection $\nabla^{(M)}$ is defined in the following way: for arbitrary vector fields $\mathbf{X},\mathbf{Y}$ tangent to the surface M, $\nabla^M_{\mathbf{X}}\mathbf{Y}$ equals to the projection on the tangent space of the vector field $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}$:

$$\nabla^M_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}},$$

where $\nabla^{\text{can.flat}}$ is canonical flat connection in $\mathbf{E}^3$ (its Christoffel symbols vanish in Cartesian coordinates). We denote by $\mathbf{A}_{\text{tangent}}$ a projection of the vector A attached at the point of the surface on the tangent space: $\mathbf{A}_\perp = \mathbf{A} - \mathbf{n}(\mathbf{A},\mathbf{n})$, ($\mathbf{n}$ is normal unit vector field to the surface.)

Theorem *Induced connection on the surface* $\mathbf{r}=\mathbf{r}(u,v)$ *in* $\mathbf{E}^3$ *coincides with Levi-Civita connection of Riemannian metric induced by the canonical metric on Euclidean space* $\mathbf{E}^3$.

Proof

Let ∇^M be induced connection on a surface M in $\mathbf{E}^3$ given by equations $\mathbf{r}=\mathbf{r}(u,v)$. Considering this connection on the basic vectors $\mathbf{r}_h, \mathbf{r}_v$ we see that it is symmetric connection. Indeed

$$\nabla^M_{\partial_u}\partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = (\mathbf{r}_{vu})_{\text{tangent}} = \nabla^M_{\partial_v}\partial_u\,. \Rightarrow \Gamma^u_{uv}=\Gamma^u_{vu}, \Gamma^v_{uv}=\Gamma^v_{vu}\,.$$

Prove that this connection preserves scalar product on M. For arbitrary tangent vector fields $\mathbf{X},\mathbf{Y},\mathbf{Z}$ we have

$$\partial_{\mathbf{X}}\langle\mathbf{Y},\mathbf{Z}\rangle_{\mathbf{E}^3} = \langle\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle_{\mathbf{E}^3} + \langle\mathbf{Y},\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\rangle_{\mathbf{E}^3}\,.$$

since canonical flat connection in $\mathbf{E}^3$ preserves Euclidean metric in $\mathbf{E}^3$ (it is evident in Cartesian coordinates). Now project the equation above on the surface M. If $\mathbf{A}$ is an arbitrary vector attached to the surface and $\mathbf{A}_{\text{tangent}}$ is its projection on the tangent space to the surface, then for every tangent vector $\mathbf{B}$ scalar product $\langle\mathbf{A},\mathbf{B}\rangle_{\mathbf{E}^3}$ equals to the scalar product $\langle\mathbf{A}_{\text{tangent}},\mathbf{B}\rangle_{\mathbf{E}^3} = \langle\mathbf{A}_{\text{tangent}},\mathbf{B}\rangle_M$ since vector $\mathbf{A}-\mathbf{A}_{\text{tangent}}$ is orthogonal to the surface. Hence we deduce from (2) that $\partial_{\mathbf{X}}\langle\mathbf{Y},\mathbf{Z}\rangle_M =$

$$\langle\left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}},\mathbf{Z}\rangle_{\mathbf{E}^3} + \langle\mathbf{Y},\left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\right)_{\text{tangent}}\rangle_{\mathbf{E}^3} = \langle\nabla^M_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle_M + \langle\mathbf{Y},\nabla^M_{\mathbf{X}}\mathbf{Z}\rangle_M\,.$$

We see that induced connection is symmetric connection which preserves the induced metric. Hence due to Levi-Civita Theorem it is unique and is expressed as in the formula (2.30).

Remark One can easy to reformulate and prove more general statement: Let M be a submanifold in Riemannian manifold (E,G). Then Levi-Civita connection of the metric induced on this submanifold coincides with the connection induced on the manifold by Levi-Civita connection of the metric G.

3 Parallel transport and geodesics

3.1 Parallel transport

3.1.1 Definition

Let M be a manifold equipped with affine connection ∇.

Definition Let $C\colon \mathbf{x}(t), t_0 \le t \le t_1$ be a curve on the manifold M, starting at the point $\mathbf{p}_0 = x(t_0)$ and ending at the point $\mathbf{p}_1 = x(_1)$ ((with coordinates $x^i = x^i(t)$)). Let $\mathbf{X} = \mathbf{X}(t_0)$ be an arbitrary tangent vector attached at the initial point $\mathbf{p}_0 = \mathbf{x}_0$ (with coordinates $x^i(t_0)$) of the curve C, i.e. $\mathbf{X}(t_0) \in T_{\mathbf{p}_0}M$ is a vector tangent to the manifold M at the point $\mathbf{p}_0$ with coordinates $x^i(t_0)$. (The vector $\mathbf{X}$ is not necessarily tangent to the curve C)

We say that $\mathbf{X}(t)$, $t_0 \le t \le t_1$ is a parallel transport of the vector $\mathbf{X}(t_0) \in T_{\partial_0}M$ along the curve $C\colon x^i = x^i(t), t_0 \le t \le t_1$ if

- For an arbitrary t, $t_0 \le t \le t$, vector $\mathbf{X} = \mathbf{X}(t)$, $(\mathbf{X}(t)|_{t=t_0} = \mathbf{X}(t_0))$ is a vector attached at the point $\mathbf{x}(t)$ of the curve C, i.e. $\mathbf{X}(t)$ is a vector tangent to the manifold M at the point $\mathbf{x}(t)$ of the curve C.

- The covariant derivative of $\mathbf{X}(t)$ along the curve C equals to zero:

$$\frac{\nabla \mathbf{X}}{dt} = \nabla_{\mathbf{v}} \mathbf{X} = 0\,. \tag{3.1}$$

  In components: if $X^m(t)$ are components of the vector field $\mathbf{X}(t)$ and $v^m(t)$ are components of the velocity vector $\mathbf{v}$ of the curve C ,

$$\mathbf{X}(t) = X^m(t)\frac{\partial}{\partial x^m}|_{\mathbf{x}(t)}\,, \quad \mathbf{v} = \frac{d\mathbf{x}(t)}{dt} = \frac{dx^i}{dt}\frac{\partial}{\partial x^m}|_{\mathbf{x}(t)}$$

  then the condition (3.1) can be rewritten as

$$\frac{dX^i(t)}{dt} + v^k(t)\Gamma^i_{km}(x^i(t))X^m(t) \equiv 0\,. \tag{3.2}$$

Remark We say sometimes that $\mathbf{X}(t)$ is *covariantly constant along the curve* C if $\mathbf{X}(t)$ is parallel transport of the vector $\mathbf{X}$ along the curve C. If we consider Euclidean space with canonical flat connection then in Cartesian coordinates Christoffel symbols vanish and parallel transport is nothing but $\frac{d\mathbf{X}}{dt} = \nabla_{\mathbf{v}}\mathbf{X} = 0$, i.e. $\mathbf{X}(t)$ is a constant vector.

3.1.2 Parallel transport is a linear map. Parallel transport with respect to Levi-Civita connection

We usualy consider parallel transport on Riemannian manifold with respect to Levi-Civita connection. If (M, G) is Riemannian manifold then we consider parallel transport with respect to connection ∇ which is Levi-Civita connection of the Riemannian metric G.

Consider again curve $C\colon \mathbf{x} = \mathbf{x}(t), t_0 \le t \le t_1$ on manifold M starting at the point $\mathbf{p}_0$ and ending at the point $\mathbf{p}_1$ (see above). Let $\mathbf{X} \in T_{\mathbf{p}_0}$ be an arbitrary tangent vector at the point $\mathbf{p}_0$, and $\mathbf{X}(t)$ be parallel transport (3.1) of this vector along the curve C:

$$\mathbf{X}(t)\big|_{t=t_0} = \mathbf{X}\,, \quad \frac{\nabla \mathbf{X}(t)}{dt} = 0\,.$$

Taking value of $\mathbf{X}(t)$ at the final point $\mathbf{p}_1$ of the curve C we come to the new vector $\mathbf{X}' = \mathbf{X}(t)\big|_{t=t_1}$ tangent to the manifold M at the point $\mathbf{p}_1$. Thus we define the map between tangent vectors at the initial point $\mathbf{p}_0$ of the curve C and tangent vectors at the ending point $\mathbf{p}_1$ of this curve:

$$P_C\colon \quad T_{\mathbf{p}_0}M \ni \mathbf{X} \longrightarrow P_C(\mathbf{X}) = \mathbf{X}' \in T_{\mathbf{p}_1}M\,. \tag{3.3}$$

Sure this map depends on the curve C which joins starting and ending points (if we are not in Euclidean space).

Proposition

Let C be a an arbitrary curve with starting point $\mathbf{p}_0$ and ending point $\mathbf{p}_1$. Then the map (3.3) defines linear operator P_C which does not depend on parameterisation of the curve, providing the initial and ending points of the curve are not swapped; i.e. operator is not changed under reparameterisations of the which do not change the orientation of the curve. (One can say that linear operator P_C is an operator defined for oriented curve, since we fix initial and ending poitns of the curve.):

$$P_C(\lambda\mathbf{X}_1 + \mu\mathbf{X}_2) = \lambda P_C(\mathbf{X}_1) + \mu P_C(\mathbf{X}_2)\,. \tag{3.4}$$

In the case if connection ∇ is Levi-Civita connection, then P_C is an orthogonal operator: for two arbitrary vectors $\mathbf{X}, \mathbf{Y} \in T_{\partial_0}M$

$$\langle \mathbf{X}, \mathbf{Y}\rangle_{\mathbf{p}_0} = \langle \mathbf{X}', \mathbf{Y}'\rangle_{\mathbf{p}_1} \quad \mathbf{X}' = P_C(\mathbf{X}) \in T_{p_1}M\,, \quad \mathbf{Y}' = P_C(\mathbf{Y}) \in T_{p_1}M\,, \tag{3.5}$$

where as usual $\langle\ ,\ \rangle_{\mathbf{p}}$ is the scalar product at the point $\mathbf{p}$.

In particular the length of the vector is preserved during parallel transport.

Proof The fact that it is a linear map follows immediately from the fact that differential equations (3.1) is linear equation. If vector fields $\mathbf{X}(t), \mathbf{Y}(t)$

are covariantly constant along the curve C, i.e. they obey differential equation (3.2), then their linear combination $\lambda\mathbf{X}(t)+\mu\mathbf{Y}(t)$ obeys this equation, also, This implies (3.4).

The fact that the map (3.3) does not depend on the parameterisation (if it does not change the orinetation) follows from differential equation (3.1) (or the same equation in components, the equation (3.2).) Indeed let $t=t(\tau)$, $\tau_0\le\tau\le\tau_1$, $t(\tau_0)=t_0, t(\tau_1)=\tau_1$ be another parameterisation of the curve C, which does not change orientation, i.e. initial and ending points of the curve do not interchange. Then multiplying the equation (3.1) on $\frac{dt}{d\tau}$ and using the fact that velocity $\mathbf{v}'(\tau)=t_\tau\mathbf{v}(t)$ we come to the differential equation in new parameterisation:

$$\frac{\nabla\mathbf{X}(t(\tau))}{d\tau}=\nabla_{\mathbf{v}'}\mathbf{X}(t(\tau))=\nabla_{t_\tau\mathbf{v}}\mathbf{X}(t(\tau))=\frac{dt}{d\tau}\nabla_{\mathbf{v}}\mathbf{X}(t)=\frac{dt}{d\tau}\frac{\nabla\mathbf{X}(t)}{dt}=0\,,$$

or in components

$$\frac{dX^i(t(\tau))}{d\tau}+v'^k(t(\tau))\Gamma^i_{km}(x^i(t(\tau)))X^m(t(\tau))\equiv 0\,. \tag{3.6}$$

The functions $X^(t(\tau))$ with the same initial conditions are the solutions of this equation.

It remains to prove that P_C is orthogonal operator.

It follows immediately from the definition (3.1) of a parallel transport and the definition (2.29) of Levi-Civita connection that during parallel transport the scalar product $\langle\mathbf{X}(t),\mathbf{Y}(t)\rangle_{\mathbf{x}(t)}$ is preserved:

$$\frac{d}{dt}\langle\mathbf{X}(t),\mathbf{Y}(t)\rangle=\partial_{\mathbf{v}}\langle\mathbf{X}(t),\mathbf{Y}(t)\rangle=\langle\nabla_{\mathbf{v}}\mathbf{X}(t),\mathbf{Y}(t)\rangle+\langle\mathbf{X}(t),\nabla_{\mathbf{v}}\mathbf{Y}(t)\rangle=\langle 0,\mathbf{Y}(t)\rangle+\langle\mathbf{X}(t),0\rangle=0\,. \tag{3.7}$$

This implies (3.5).

3.2 Geodesics

3.2.1 Definition. Geodesics on Riemannian manifold

We are going to define geodesics in Riemannian manifold.

Geodesic is generalisation of straight line.

Straight line is

- the shortest

- the straightest: i.e. the velocity vector at all the points is the 'same'
- trajectory of free particle

Any of these properties may be generalised. We will focus attention on the second one.

Let M be Riemannian manifold equipped with Levi-Civita connection ∇.

Definition A parameterised curve $C\colon \quad x^i = x^i(t)$ in Rieamannian manifold is called geodesic if velocity vector $\mathbf{v}(t)\colon v^i(t) = \frac{dx^i(t)}{dt}$ is covariantly constant along this curve, i.e. parallel transport of velocity vector along the curve preserves the velocity vector:

$$\nabla_{\mathbf{v}}\mathbf{v} = \frac{\nabla \mathbf{v}}{dt} = \frac{dv^i(t)}{dt} + v^k(t)\Gamma^i_{km}(x(t))v^m(t) = 0, \quad i.e. \tag{3.8}$$

$$\frac{d^2x^i(t)}{dt^2} + \frac{dx^k(t)}{dt}\Gamma^i_{km}(x(t))\frac{dx^m(t)}{dt} = 0\,. \tag{3.9}$$

These are linear second order differential equations. One can prove that this equations have solution and it is unique[10] for an arbitrary initial data $(x^i(t_0) = x^i_0, \dot{x}^i(t_0) = \dot{x}^i_0.\ )$

In other words the curve $C\colon x(t)$ is a geodesic if parallel transport of velocity vector along the curve is a velocity vector at any point of the curve.

Remark One can see that our definition of geodesic works for arbitrary connection. However we will consider here only geodesics on Riemannian manifold, defined only with Levi-Civita connection.

Since velocity vector of the geodesics on Riemannian manifold at any point is a parallel transport with the Levi-Civita connection, hence due to Proposition above (see equation (3.4), (3.5) and (3.7)) the length of the velocity vector remains constant:

Proposition *If* $C\colon \mathbf{x}(t)$ *is a geodesics on Riemannian manifold then the length of velocity vector is preserved along the geodesic.*

Proof Since the connection is Levi-Civita connection then it preserves scalar product of tangent vectors, (see (2.29)) in particularly the length of the velocity vector $\mathbf{v}$:

[10] this is true under additional technical conditions which we do not discuss here

Example 1 *Geodesics of Euclidean space.* In Cartesian coordinates Christoffel symbols of Levi-Civita connection vanish, and differential equation (3.8), (3.9) are reduced to equation

$$\frac{d^2x^i(t)}{dt^2} = 0\,, \Rightarrow \frac{dx^i(t)}{dt} = v^i \Rightarrow x^i = x^i_0 + v^i t\,. \tag{3.10}$$

We come to straight lines.

Example 2 *Geodesics of cylindrical surface* One can see that if Riemannian metric $G = G_{ik} du^i dv^k$ have constant coefficients in coordinates u^i then Christoffel symbols of Levi-Civita connection vanish in these coordinates, (see formula (2.30)) and according to (3.10) geodesics are "straight lines" in coordinates u^i. In particular this is a case for cylinder: If surface of cylinder is given by equation $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$ then Riemannian metric is equal to $G = a^2 d\varphi^2 + dh^2$ and we come to equations:

$$\begin{cases} \frac{d^2\varphi(t)}{dt^2} = 0 \\ \frac{d^2h(t)}{dt^2} = 0 \end{cases} \Rightarrow \begin{cases} \frac{d\varphi(t)}{dt} = \Omega \\ \frac{dh(t)}{dt} = c \end{cases} \Rightarrow \begin{cases} \varphi(t) = \varphi_0 + \Omega t \\ h(t) = h_0 + tc \end{cases}. \tag{3.11}$$

In general case we come to helix:

$$\begin{cases} x = a\cos\varphi(t) = a\cos(\varphi_0 + \Omega t) \\ y = a\sin\varphi(t) = a\sin(\varphi_0 + \Omega t) \\ z = h(t) = h_0 + ct \end{cases} \tag{3.12}$$

If $c = 0$ then geodesics are circles $x^2 + y^2 = a^2, z = h_0$. If angular velocity $\Omega = 0$ then geodesics are vertical lines $x = x_0, y = y_0, z = h_0 + ct$.

3.2.2 Geodesics and Lagrangians of "free" particle on Riemannian manifold.

Lagrangian and Euler-Lagrange equations

A function $L = L(x, \dot{x})$ on points and velocity vectors on manifold M is a *Lagrangian* on manifold M.

We assign to Lagrangian $L = L(x, \dot{x})$ the following second order differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \tag{3.13}$$

In detail
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial^2 L}{\partial x^m \partial \dot{x}^i}\dot{x}^m + \frac{\partial^2 L}{\partial \dot{x}^m \partial \dot{x}^i}\overset{..}{x}^m = \frac{\partial L}{\partial x^i}\,. \tag{3.14}$$

These equations are called *Euler-Lagrange equations* of the Lagrangian L. E.g. for particle in external field with potential energy $U = U(\mathbf{r})$ in Cartesian coordinates in $\mathbf{E}^n$,

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \text{kinetic energy} - \text{potential energy} = \frac{m\mathbf{v}^2}{2} - U(\mathbf{r})\,. \tag{3.15}$$

We will explain later the variational origin of these equations

Remark To every mechanical system one can put in correspondence a Lagrangian on configuration space. The dynamics of the system is described by Euler-Lagrange equations. The advantage of Lagrangian approach is that it works in an arbitrary coordinate system: Euler-Lagrange equations are invariant with respect to changing of coordinates since they arise from variational principe. E.g. one can easy to see that Euler-Lagrange equations

$$m\frac{d\mathbf{v}(t)}{dt} = -\frac{\partial U(\mathbf{r})}{\partial \mathbf{r}_i} \tag{3.16}$$

for Lagrangian (3.15) coincide with standard Newton equations (3.16) in Cartesian coordinates. However Newton equations (3.16) do not survive under arbitrary coordinate transformations contrary to Euler-Lagrange equations.

Lagrangian of "free" particle

Let (M, G), $G = g_{ik}dx^i dx^k$ be a Riemannian manifold.

Definition We say that *Lagrangian* $L = L(x, \dot{x})$ is the Lagrangian of a a 'free' particle on the Riemannian manifold M if

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} \tag{3.17}$$

Example "Free" particle in Euclidean space. Consider $\mathbf{E}^3$ with standard metric $G = dx^2 + dy^2 + dz^2$

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} \tag{3.18}$$

Note that this is the Lagrangian that describes the dynamics of a free particle.

Example A 'free' particle on a sphere.

The metric on the sphere of radius R is $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Respectively for the Lagrangian of "free" particle we have

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} = \frac{R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2}{2} \tag{3.19}$$

Equations of geodesics and Euler-Lagrange equations

Theorem. *Euler-Lagrange equations of the Lagrangian of a free particle are equivalent to the second order differential equations for geodesics.*

This Theorem makes very easy calculations for Christoffel indices.

This Theorem can be proved by direct calculations.

Calculate Euler-Lagrange equations (3.2.2) for the Lagrangian (3.17):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(\frac{\partial\left(\frac{g_{mk}\dot{x}^m\dot{x}^k}{2}\right)}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(g_{ik}\dot{x}^k\right) = g_{ik}\ddot{x}^{\,k} + \frac{\partial g_{ik}}{\partial x^m}\dot{x}^m\dot{x}^k$$

and

$$\frac{\partial L}{\partial x^i} = \frac{\partial\left(\frac{g_{mk}\dot{x}^m\dot{x}^k}{2}\right)}{\partial x^i} = \frac{1}{2}\frac{\partial g_{mk}}{\partial x^i}\dot{x}^m\dot{x}^k.$$

Hence we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = g_{ik}\ddot{x}^{\,k} + \frac{\partial g_{ik}}{\partial x^m}\dot{x}^m\dot{x}^k = \frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial g_{mk}}{\partial x^i}\dot{x}^m\dot{x}^k\,,$$

i.e.

$$g_{ik}\ddot{x}^{\,k} + \partial_m g_{ik}\dot{x}^m\dot{x}^k = \frac{1}{2}\partial_i g_{mk}\dot{x}^m\dot{x}^k\,.$$

Note that $\partial_m g_{ik}\dot{x}^m\dot{x}^k = \frac{1}{2}\left(\partial_m g_{ik}\dot{x}^m\dot{x}^k + \partial_k g_{im}\dot{x}^m\dot{x}^k\right)$. Hence we come to equation:

$$g_{ik}\frac{d^2x^k}{dt^2} + \frac{1}{2}\left(\partial_m g_{ik} + \partial_k g_{im} - \partial_i g_{mk}\right)\dot{x}^m\dot{x}^k$$

Multiplying on the inverse matrix g^{ik} we come

$$\frac{d^2x^i}{dt^2} + \frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)\frac{dx^m}{dt}\frac{dx^k}{dt} = 0\,. \tag{3.20}$$

We recognize here Christoffel symbols of Levi-Civita connection (see (2.30)) and we rewrite this equation as

$$\frac{d^2x^i}{dt^2} + \frac{dx^m}{dt}\Gamma^i_{mk}\frac{dx^k}{dt} = 0\,. \tag{3.21}$$

This is nothing but the equation (3.8).

Applications of this Theorem: calculation of Christoffel symbols of Levi-Civita connection.

3.2.3 Calculations of Christoffel symbols and geodesics using the Lagrangians of a free particle.

It turns out that equation (3.21) is the very effective tool to caluclatie Christoffel symbols of Levi-Civita connection.

Examples in this subsection will be calculated in detail on tutorial (see Homework 7)

Consider two examples: We calculate Levi-Civita connection on sphere in $\mathbf{E}^3$ and on Lobachevsky plane using Lagrangians and find geodesics.

1) *Sphere of the radius R in* $\mathbf{E}^3$:

Lagrangian of "free" particle on the sphere is given by (3.19):

$$L = \frac{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}{2}$$

Euler-Lagrange equations defining geodesics are

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(R^2\dot\theta\right) - R^2\sin\theta\cos\theta\dot\varphi^2 \Rightarrow \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0\,,\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \frac{\partial L}{\partial\varphi} = \frac{d}{dt}\left(R^2\sin^2\theta\dot\varphi\right) = 0 \Rightarrow \ddot\varphi + 2\mathrm{cotan}\,\theta\dot\theta\dot\varphi = 0\,.\end{gathered} \tag{3.22}$$

Comparing Euler-Lagrange equations with equations for geodesic in terms of Christoffel symbols:

$$\ddot\theta + \Gamma^\theta_{\theta\theta}\dot\theta^2 + 2\Gamma^\theta_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\theta_{\varphi\varphi}\dot\varphi^2 = 0,$$

$$\ddot\varphi + \Gamma^\varphi_{\theta\theta}\dot\theta^2 + 2\Gamma^\varphi_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi^2 = 0$$

we come to

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0\,, \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\,, \tag{3.23}$$

$$\Gamma^{\varphi}_{\theta\theta} = \Gamma^{\varphi}_{\varphi\varphi} = 0,\ \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\!\mathrm{an}\,\theta\,. \tag{3.24}$$

(Compare with previous calculations for connection in subsections 2.2.1 and 2.3.4)

We know already and we will prove later in a elegant way that geodesics on the sphere are great circles. (see subsection 3.2.6 below). Consider another technically more difficult but straightforward proof of this fact. To find geodesics one have to solve second order differential equations (3.21)

One can see that the great circles: $\varphi = \varphi_0$, $\theta = \theta_0 + t$ are solutions of second order differential equations (3.22) with initial conditions

$$\theta(t)\big|_{t=0} = \theta_0, \dot\theta(t)\big|_{t=0} = 1, \quad \varphi(t)\big|_{t=0} = \varphi_0, \dot\theta(t)\big|_{t=0} = 0\,. \tag{3.25}$$

The rotation of the sphere is isometry, which does not change Levi-Civta connection. Hence an arbitrary great circle is geodesic.

Prove that an arbitrary geodesic is an arc of great circle. Let the curve $\theta = \theta(t), \varphi = \varphi(t)$, $0 \leq t \leq t_1$ be geodesic. Rotating the sphere we can come to the curve $\theta = \theta'(t), \varphi = \varphi'(t)$, $0 \leq t \leq t_1$ such that velocity vector at the initial time is direccted along meridian, i.e. initial conditions are

$$\theta'(t)\big|_{t=0} = \theta_0, \dot\theta'(t)\big|_{t=0} = a, \quad \varphi'(t)\big|_{t=0} = \varphi_0, \dot\varphi'(t)\big|_{t=0} = 0\,. \tag{3.26}$$

(Compare with initial conditions (3.25)) Second order differential equations with boundary conditions for coordinates and velocities at $t = 0$ have unique solution. The solutions of second order differential equations (3.22) with initial conditions (3.26) is a curve $\theta'(t) = \theta_0 + at$, $\varphi'(t) = \varphi_0$. It is great circle. Hence initial curve the geodesic $\theta = \theta(t), \varphi = \varphi(t)$, $0 \leq t \leq t_1$ is an arc of great circle too.

This is another proof that geodesics are great circles.

2) *Lobachevsky plane.*

Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2+dy^2}{y^2}$ is

$$L = \frac{1}{2}\frac{\dot x^2 + \dot y^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot x} = \frac{d}{dt}\left(\frac{\dot x}{y^2}\right) = \frac{\ddot x}{y^2} - \frac{2\dot x\dot y}{y^3}, \text{i.e.} \quad \ddot x - \frac{2\dot x\dot y}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot x^2 + \dot y^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial \dot y} = \frac{d}{dt}\left(\frac{\dot y}{y^2}\right) = \frac{\ddot y}{y^2} - \frac{2\dot y^2}{y^3}, \text{i.e.} \quad \ddot y + \frac{\dot x^2}{y} - \frac{\dot y^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot{x}^i - \dot{x}^k \Gamma^i_{km} \dot{x}^m = 0$ $(i = 1, 2,\ x = x^1, y = x^2)$ we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y},\ \Gamma^x_{yy} = 0,\ \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}\,. \blacksquare$$

In a similar way as for a sphere one can find geodesics on Lobachevsky plane. First we note that vertical rays are geodesics. Then using the inversions with centre on the absolute one can see that arcs of the circles with centre at the absolute ($y = 0$) are geodesics too.

See also examples in Homework 6

3.2.4 Un-parameterised geodesic

We defined a geodesic as a parameterised curve such that the velocity vector is covariantly constant along the curve.

What happens if we change the parameterisation of the curve?

Another question: Suppose a tangent vector to the curve remains tangent to the curve during parallel transport. Is it true that this curve (in a suitable parameterisation) becomes geodesic?

Definition We call un-parameterised curve geodesic if under suitable parameterisation it obeys the equation (3.8) for geodesics.

Let C–be un-parameterised geodesic. Then the following statement is valid.

Proposition *A curve C (un-parameterised) is geodesic if an only if a non-zero vector tangent to the curve remains tangent to the curve during parallel transport.*

Proof. Let $\mathbf{A}$ be tangent vector at the point $\mathbf{p} \in C$ of the curve. Parallel transport does not depend on parametersiation of the curve (see subsection 3.1.2, equation (3.3)). Choose a suitable parameterisation $x^i = x^i(t)$ such that $x^i(t)$ obeys the equations (3.8) for geodesics, i.e. the velocity vector $\mathbf{v}(t)$ is covariantly constant along the curve: $\nabla_{\mathbf{v}}\mathbf{v} = 0$. If $\mathbf{A}(t_0) = c\mathbf{v}(t_0)$ at the given point $\mathbf{p}$ (c is a scalar coefficient) then due to linearity $\mathbf{A}(t) = c\mathbf{v}(t)$ is a parallel transport of the vector $\mathbf{A}$. The vector $\mathbf{A}(t)$ is tangent to the curve since it is proportional to velocity vector. We proved that any tangent vector remains tangent during parallel transport.

Now prove the converse: Let $\mathbf{A}(t)$ be a parallel transport of non-zero vector and it is proportional to velocity, i.e. $\mathbf{A}(t) = c(t)\mathbf{v}(t)$. Thus

$$\frac{\nabla \mathbf{A}(t)}{dt} = \nabla_{\mathbf{v}}\mathbf{A} = 0\,,$$

Choose a reparameterisation $t = t(\tau)$ such that $\frac{dt(\tau)}{d\tau} = c(t)$. In the new parameterisation the velocity vector $\mathbf{v}'(\tau) = \frac{dt(\tau)}{d\tau}\mathbf{v}(t(\tau)) = c(t)\mathbf{v}(t) = \mathbf{A}(t(\tau))$ and

$$\frac{d\mathbf{v}'}{d\tau} = \nabla_{\mathbf{v}'}\mathbf{v}' = \nabla_{t_\tau \mathbf{v}}(t_\tau \mathbf{v}) = t_\tau \nabla_{\mathbf{v}}(t_\tau \mathbf{v}) = t_\tau \nabla_{\mathbf{v}}(c\mathbf{v}) = t_\tau \nabla_{\mathbf{v}}\mathbf{A} = 0\,.$$

We come to parameterisation such that velocity vector remains covariantly constant, i.e. it is parameterised geodesic. (This is reparameterisation invariance of parallel transport (3.1) (see (3.6)). Thus we come to parameterised geodesic. Hence C is a geodesic.

Remark In particularly it follows from the Proposition above the following important observation:

Let C is un-parameterised geodesic, $x^i(t)$ be its arbitrary parameterisation and $\mathbf{v}(t)$ be velocity vector in this parameterisation. Then the velocity vector remains parallel to the curve since it is a tangent vector.

In spite of the fact that velocity vector is not covariantly constant along the curve, i.e. it will not remain velocity vector during parallel transport, since it will be remain tangent to the curve during parallel transport.

Remark One can see that if $x^i = x^i(t)$ is geodesic in an arbitrary parameterisation and $s = s(t)$ is a natural parameter (which defines the length of the curve) then $x^i(t(s))$ is parameterised geodesic.

3.2.5 Parallel transport of vectors along geodesics

We already now that during parallel transport along curve with respect to Levi-Civita connection scalar product of vectors, i.e. lengths of vectors and angle between them does not change (see subsection 3.1.2, equations (3.3) and (3.5)). This remark makes easy to calculate parallel transport of vectors along geodesics in Riemannian manifold. Indeed let C a geodesic (in general un-parameterised) and a vectors $\mathbf{X}(t)$ is attached to the point $\mathbf{p}_1 \in C$ on the curve C. In the special case if $\mathbf{X}$ is a tangent vector to geodesic C then during parallel transport it remains tangent, i.e. proportional to velocity vector:

$$\mathbf{X}(t) = a(t)\mathbf{v}(t)\,. \tag{3.27}$$

Here $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ and $\mathbf{r} = \mathbf{r}(t)$ is an *arbitrary parameterisation* of geodesic C. Note that in general t is not parameter such that $\mathbf{r} = \mathbf{r}(t)$ is parameterised geodesic; t is an arbitrary parameter. In the special case if t is a parameter such that $\mathbf{r} = \mathbf{r}(t)$ is parameterised geodesic then velocity vector remains velocity vector during parallel transport, i.e. $\mathbf{X}(t) = a\mathbf{v}(t)$ where a is not dependent on t.

To calculate the dependence of coefficient a on t in (3.28) we note that the length of the vector is not changed (see equation (3.5) in the section 3.1.2, i.e.

$$\langle \mathbf{X}(t), \mathbf{X}(t)\rangle = \langle a(t)\mathbf{v}(t), a(t)\mathbf{v}(t)\rangle = a^2(t)|\mathbf{v}(t)|^2 = \text{constant} \tag{3.28}$$

3.2.6 Geodesics on surfaces in $\mathbf{E}^3$

Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$. Let G_M be induced Riemannian metric and ∇ a Levi-Civita connection on M. We consider on M Levi-Civita connection of the metric G_M.

Let C be an arbitrary geodesic and $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ the velocity vector. According to the definition of geodesic $\nabla_{\mathbf{v}}\mathbf{v} = 0$. On the other hand we know that Levi-Civita connection coincides with the connection induced on the surface by canonical flat connection in $\mathbf{E}^3$ (see the Theorem in subsection 2.4). Hence

$$\nabla_{\mathbf{v}}\mathbf{v} = 0 = \nabla^M_{\mathbf{v}}\mathbf{v} = \left(\nabla^{\text{can.flat}}_{\mathbf{v}}\mathbf{v}\right)_{\text{tangent}} \tag{3.29}$$

In Cartesian coordinates $\nabla^{\text{can.flat}}_{\mathbf{v}}\mathbf{v} = \partial_{\mathbf{v}}\mathbf{v} = \frac{d}{dt}\mathbf{v}(u(t), v(t)) = \frac{d^2\mathbf{r}(t)}{dt^2} = \mathbf{a}$.

Hence according to (3.29) the tangent component of acceleration equals to zero.

Converse if for the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ the acceleration vector $\mathbf{a}(t)$ is orthogonal to the surface then due to (3.29) $\nabla_{\mathbf{v}}\mathbf{v} = 0$.

We come to very beautiful observation:

Theorem *The acceleration vector of an curve $\mathbf{r} = \mathbf{r}(u(t), v(t))$ on M is orthogonal to the surface M if and only if this curve is geodesic.*

In other words due to Newton second law particle moves along along geodesic on the surface if and only if the force is orthogonal to the surface.

One can very easy using this Proposition to calculate geodesics of cylinder and sphere.

Geodesic on the cylinder

Let $\mathbf{r}(h(t), \varphi(t))$ be a geodesic on the cylinder $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$. We have $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} -a\dot\varphi\sin\varphi \\ a\dot\varphi\cos\varphi \\ \dot h \end{pmatrix}$ and for acceleration:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \underbrace{\begin{pmatrix} -a\ddot\varphi\,\sin\varphi \\ a\ddot\varphi\,\cos\varphi \\ \ddot h \end{pmatrix}}_{\text{tangent acceleration}} + \underbrace{\begin{pmatrix} -a\dot\varphi^2\cos\varphi \\ -a\dot\varphi^2\sin\varphi \\ 0 \end{pmatrix}}_{\text{normal acceleration}}$$

Since tangential acceleration equals to zero hence $\frac{d^2\varphi}{dt^2} = 0$, $\varphi(t) = \varphi_0 + \Omega t$, and $\frac{d^2h}{dt^2} = 0$, $h(t) = h_0 + ct$. Normal acceleration is centripetal acceleration of the rotation over circle with constant speed (projection on the plane OXY). The geodesic is helix. (Compare these calculations with calculations of geodesics of cylinder in the last example of section 3.2.1: see (3.12).)

Geodesics on sphere

Let $\mathbf{r} = \mathbf{r}(\theta(t), \varphi(t))$ be a geodesic on the sphere of the radius a: $\mathbf{r}(\theta,\varphi)\colon \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases}$

Consider the vector product of the vectors $\mathbf{r}(t)$ and velocity vector $\mathbf{v}(t)$ $\mathbf{M}(t) = \mathbf{r}(t) \times \mathbf{v}(t)$. Acceleration vector $\mathbf{a}(t)$ is proportional to the $\mathbf{r}(t)$ since due to Proposition it is orthogonal to the surface of the sphere. This implies that $\mathcal{M}(t)$ is constant vector:

$$\frac{d}{dt}\mathbf{M}(t) = \frac{d}{dt}\left(\mathbf{r}(t) \times \mathbf{v}(t)\right) = \left(\mathbf{v}(t) \times \mathbf{v}(t)\right) + \left(\mathbf{r}(t) \times \mathbf{a}(t)\right) = 0 \qquad (3.30)$$

We have $\mathbf{M}(t) = \mathbf{M}_0$. $\mathbf{r}(t)$ is orthogonal to $\mathbf{M} = \mathbf{r}(t) \times \mathbf{v}(t)$. We see that $\mathbf{r}(t)$ belongs to the sphere and to the plane orthogonal to the vector $\mathbf{M}_0 = \mathbf{r}(t) \times \mathbf{v}(t)$. The intersection of this plane with sphere is a great circle. We proved that if $\mathbf{r}(t)$ is geodesic hence it belongs to great circle (as un-parameterised curve).

The converse is evident since if particle moves along the great circle with constant velocity then obviously acceleration vector is orthogonal to the surface.

Remark The vector $\mathcal{M} = \mathbf{r}(t) \times \mathbf{v}(t)$ is the torque. The torque is integral of motion in isotropic space.—This is the core of the considerations for geodesics on the sphere.

3.2.7 Geodesics and shortest distance.

Many of you know that geodesics are in some sense shortest curves. We will give here an exact meaning to this statement The proof is using variational principe. (See the proof, discussions and applications of this statement in appendices.)

Let M be a Riemannian manifold.

Theorem *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points on M. The shortest curve which joins these points is an arc of geodesic.*

Let C be a geodesic on M and $\mathbf{x}_1 \in C$. Then for an arbitrary point $\mathbf{x}_2 \in C$ which is close to the point $\mathbf{x}_1$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points[11] .

This Theorem makes a bridge between two different approach to geodesic: the shortest distance and parallel transport of velocity vector.

4 Surfaces in $\mathbf{E}^3$. Parallel transport of vectors and *Theorema Egregium*

Now equipped by the knowledge of Riemannian geometry we consider surfaces in $\mathbf{E}^3$, and formulate the important theorem about parallel transport of vector over closed curve on the surface. As an important corollary of this Theorem we will formulate and prove Gauß *Theorema Egregim*

[11]More precisely: for every point $\mathbf{x}_1 \in C$ there exists a ball $B_\delta(\mathbf{x}_1)$ such that for an arbitrary point $\mathbf{x}_2 \in C \cap B_\delta(\mathbf{x}_1)$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points.

4.1 Parallel transport of the vector and Gaussian curvature of surface.

We formulate here very important theorem about parallel transport of vectors over closed curve and deduce Theorema Egregium from this theorem.

4.1.1 Theorem of parallel transport over closed curve. Preliminary formulation

We give preliminary formulation of this Theorem. Later when we will learn Gaussian curvature we will return again to this theorem (see 4.1.5 above.)

Let M be a surface in Euclidean space $\mathbf{E}^3$. Consider a closed curve C on M, $M\colon \mathbf{r} = \mathbf{r}(u,v)$, $C\colon \mathbf{r} = \mathbf{r}(u(t, v(t)), 0 \le t \le t_1$, $\mathbf{x}(0) = \mathbf{x}(t_1)$. $(u(t), v(t)$ are internal coordinates of the curve C.)

Consider the parallel transport of an arbitrary tangent $\mathbf{X}$ vector along the closed curve C:

$$\mathbf{X}(t)\colon \ \frac{\nabla \mathbf{X}(t)}{dt} = 0, \ 0 \le t \le t_1 \,,$$

i.e.

$$\frac{dX^\alpha(t)}{dt} + X^\beta(t)\Gamma^\alpha_{\beta\gamma}(u(t))\frac{du^\gamma(t)}{dt} = 0, \ 0 \le t \le t_1 \,, \qquad \alpha, \beta, \gamma = 1, 2 \,, \quad (4.1)$$

where ∇ is the connection induced on the surface M by the Levi-Civita connection (2.30) of the induced Riemannian metric on the surface M, (this is the same as to say that ∇ is the connection induced on the surface M by canonical flat connection (see (2.23))), and $\Gamma^\alpha_{\beta\gamma}$ its Christoffel symbols (see in more detail in subsection 4.1.5 above).

Theorem *Let M be a surface in Euclidean space $\mathbf{E}^3$. Let C be a closed curve C on M such that C is a boundary of a compact oriented domain $D \subset M$. Consider the parallel transport of an arbitrary tangent vector along the closed curve C. As a result of parallel transport along this closed curve any tangent vector rotates through the angle*

$$\angle\phi = \angle\left(\mathbf{X}, P_C\mathbf{X}\right) = \int_D K d\sigma \,, \qquad (4.2)$$

where K is the Gaussian curvature and $d\sigma$ is the area element of induced Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g}dudv$, where $g_{\alpha\beta} = (\mathbf{r}_\alpha, \mathbf{r}_\beta)$.

Example *Consider the closed curve, "latitude" C_{θ_0}: $\theta=\theta_0$ on the sphere of the radius R. Calculations show that*

$$\angle\phi(C_{\theta_0}) = 2\pi(1-\cos\theta_0) \tag{4.3}$$

(see also the Homework 8). On the other hand the latitude C_{θ_0} is the boundary of the segment D with area $2\pi RH$ where $H = R(1-\cos\theta_0)$. Hence

$$\angle\left(\mathbf{X}, \mathbf{R}_C\mathbf{X}\right) = \frac{2\pi RH}{R^2} = \frac{1}{R^2} \cdot \text{area of the segment} = \int_D K d\sigma$$

since Gaussian curvature is equal to $\frac{1}{R^2}$

In the statement of this Theorem we use the Gaussian curvature. We will explain it in next subsections, then will return again to this Theorem.

4.1.2 Weingarten (shape) operator on surfaces and Gaussian curvature

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface and $\mathbf{n}(u,v)$ be a unit normal vector field at the points of the surface M.

We define at every point $\mathbf{p} = \mathbf{r}(u,v)$ the Weingarten (shape) operator S acting on the vector space $T_\mathbf{p}M$ of vectors tangent to the surface M.

Definition-Proposition *Let $\mathbf{n}(u,v)$ be a unit normal vector field to the surface M. Then operator*

$$S\colon\ S(\mathbf{X}) = \partial_\mathbf{X}(-\mathbf{n}) = -X_u\frac{\partial \mathbf{n}(u,v)}{\partial u} - X_v\frac{\partial \mathbf{n}(u,v)}{\partial v} \tag{4.4}$$

maps tangent vectors to the tangent vectors:

$$S\colon T_\mathbf{p}M \to T_\mathbf{p}M \text{ for every } \mathbf{X} = X_u\mathbf{r}_u + X_v\mathbf{r}_v \in T_\mathbf{p}M, \qquad S(\mathbf{X}) \in T_\mathbf{p}M \tag{4.5}$$

This operator is called Weingarten (shape) operator.

Remark *The sign "$-$" seems to be senseless: if $\mathbf{n}$ is unit normal vector field then $-\mathbf{n}$ is normal vector field too. Later we will see why it is convenient (see the Example-Motivation and Proposition below).*

Show that property (4.5) is indeed obeyed, i.e. vector $\mathbf{X}' = S(\mathbf{X})$ is tangent to surface. Consider derivative of scalar product $(\mathbf{n},\mathbf{n})$ with respect to the vector field $\mathbf{X}$. We have that $(\mathbf{n},\mathbf{n}) = 1$. Hence

$$\partial_\mathbf{X}(\mathbf{n},\mathbf{n}) = 0 = \partial_\mathbf{X}(\mathbf{n},\mathbf{n}) = (\partial_\mathbf{X}\mathbf{n},\mathbf{n}) + (\mathbf{n},\partial_\mathbf{X}\mathbf{n}) = 2(\partial_\mathbf{X}\mathbf{n},\mathbf{n})\,.$$

Hence $(\partial_{\mathbf{X}}\mathbf{n},\mathbf{n}) = -(S(\mathbf{X}),\mathbf{n}) = -(\mathbf{X}',\mathbf{n}) = 0$, *i.e. vector* $\partial_{\mathbf{X}}\mathbf{n} = -\mathbf{X}'$ *is orthogonal to the vector* $\mathbf{n}$. *This means that vector* $\mathbf{X}'$ *is tangent to the surface.*

Write down the action of shape operator on coordinate basis $\mathbf{r}_u = \partial_u$, $\partial_v = \mathbf{r}_v$ *at the given point* $\mathbf{p}$*:*

$$S(\mathbf{r}_u) = -\partial_{\mathbf{r}_u}\mathbf{n}(u,v) = -\frac{\partial\mathbf{n}(u,v)}{\partial u}, \quad S(\mathbf{r}_v) = -\partial_{\mathbf{r}_v}\mathbf{n}(u,v) = -\frac{\partial\mathbf{n}(u,v)}{\partial v}$$

Since the shape operator transforms tangent vectors to tangent vectors, then

$$\begin{array}{c} S(\mathbf{r}_u) = -\frac{\partial\mathbf{n}(u,v)}{\partial u} = a\,\mathbf{r}_u + c\mathbf{r}_v \\ S(\mathbf{r}_v) = -\frac{\partial\mathbf{n}(u,v)}{\partial v} = b\mathbf{r}_u + d\mathbf{r}_v \end{array},$$

i.e.

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in the coordinate basis } \mathbf{r}_u, \mathbf{r}_v \tag{4.6}$$

(This matrix is the matrix of Weingarten operator in the basis $(\mathbf{r}_u, \mathbf{r}_v)$*.)*

Remark. *Shape operator as well as normal unit vector is defined up to a sign:*

$$\mathbf{n}(u,v) \to -\mathbf{n}(u,v), \quad \text{then} \quad S \to -S\,.$$

Example-Motivation *Consider just for curve* C *in* $\mathbf{E}^2$. *analog of shape operator. Let* $\mathbf{r} = \mathbf{r}(t)$ *be parameterised curve, and* $\mathbf{n} = \mathbf{n}(t)$ *be unit normal vector field on the curve, then for arbitrary tangent vector* $\mathbf{x} = c\mathbf{v}$ *(where* $\mathbf{v}$ *is a velocity vector)*

$$T_{\mathbf{p}}C \ni \mathbf{x} = c\mathbf{v} \to S(\mathbf{x}) = -\partial_{\mathbf{x}}\mathbf{n} = -c\frac{dx^i}{dt}\frac{\partial\mathbf{n}(\mathbf{r}(t))}{\partial x^i} = --\frac{d\mathbf{n}(\mathbf{r}(t))}{dt} = k(\mathbf{r}(t))\mathbf{v}$$

since vector field $\frac{d\mathbf{n}(\mathbf{r}(t))}{dt}$ *is tangent to the curve. Here* k *is so called curvature of the curve (Frenet curvature)(Usually curvature is defined as modulus of this magnitude).*

Explain how curvature is related with normal acceleration (centripetal acceleration) $\mathbf{a}_{\text{normal}}$. *We have:*

$$\mathbf{a}_{\text{normal}} = (\mathbf{a},\mathbf{n})\mathbf{n} = \left(\frac{d\mathbf{v}}{dt},\mathbf{n}\right)\mathbf{n} = -\left(\mathbf{v},\frac{d\mathbf{n}}{dt}\right)\mathbf{n} = (\mathbf{v},S(\mathbf{v}))\,\mathbf{n}$$

(you see this equation explains why the sign "$-$" appears)

Note that it follows from this equation that if $\mathbf{v}(t)$ is tangent unit vector field, then

$$\frac{d\mathbf{v}(t)}{dt} = k(t)\mathbf{n}$$

it is nothing but centripetal acceleration.

Curvature of curves is not intrincis object.

We show now that normal acceleration of a curve on the surface and normal curvature are expressed in terms of shape operator.

Let $C\colon \mathbf{r}(t)$ be a curve on the surface M, $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$. Let $\mathbf{v} = \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a} = \mathbf{a}(t) = \frac{d^2 r(t)}{dt^2}$ be velocity and acceleration vectors respectively. Recall that

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z = \frac{d\mathbf{r}(u(t), v(t))}{dt} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v \tag{4.7}$$

be velocity vector; $\dot{u}, \dot{v}$ are internal components of the velocity vector with respect to the basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and $\dot{x}, \dot{y}, \dot{z}$, are external components velocity vectors with respect to the basis $\{\mathbf{e}_x = \partial_x, \mathbf{e}_y = \partial_y, \mathbf{e}_z = \partial_z\}$. As always we denote by $\mathbf{n}$ normal unit vector.

Proposition *The normal acceleration at an arbitrary point $\mathbf{p} = \mathbf{r}(u(t_0), v(t_0))$ of the curve C on the surface M is defined by the scalar product of the velocity vector $\mathbf{v}$ of the curve at the point $\mathbf{p}$ on the value of the shape operator on the velocity vector:*

$$\mathbf{a}_n = a_n\mathbf{n} = (\mathbf{v}, S\mathbf{v})\,\mathbf{n} \tag{4.8}$$

and normal curvature is equal to

$$\kappa_n = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v}, \mathbf{v})} \tag{4.9}$$

Proof of the Proposition. According to we have

$$\mathbf{a}_n = (\mathbf{n}, \mathbf{a})\mathbf{n} = \mathbf{n}\left(\mathbf{n}, \frac{d}{dt}\mathbf{v}(t)\right)\mathbf{n} = \mathbf{n}\frac{d}{dt}\left(\mathbf{n}, \mathbf{v}(t)\right) - \mathbf{n}\left(\frac{d}{dt}\mathbf{n}(u(t), v(t)), \mathbf{v}(t)\right)$$

$$= 0 + \left(-\partial_{\mathbf{v}}\mathbf{n}, \mathbf{v}\right)\mathbf{n} = (S\mathbf{v}, \mathbf{v})\mathbf{n}$$

This proves Proposition.

Later we will use equation (4.9) to find eigenvectors of the operator.

4.1.3 The Weingarten operator; principal curvatures and Gaussian curvature

Now we introduce on surfaces, principal curvatures and Gaussian curvature in terms of the Weingarten (shape) operator

Let $\mathbf{p}$ *be an arbitrary point of the surface* M *and* S *be the Weingarten operator at this point.* S *is symmetric operator:* $(S\mathbf{a},\mathbf{b}) = (\mathbf{b}, S\mathbf{a})$. *Consider eigenvalues* λ_1, λ_2 *and eigenvectors* $\mathbf{l}_1, \mathbf{l}_2$ *of the shape operator* S

$$\mathbf{l}_1, \mathbf{l}_2 \in T_{\mathbf{p}}M, \qquad S\,\mathbf{l}_1 = \kappa_1 l_1, \qquad S\,\mathbf{l}_2 = \kappa_2 l_2, \tag{4.10}$$

Definition *Eigenvalues of shape operator* λ_1, λ_2 *are called principal curvatures:*

$$\lambda_1 = \kappa_1, \;\; \lambda_2 = \kappa_2$$

Eigenvectors λ_1, λ_2 *define the two directions such that curves directed along these vectors have normal curvature equal to the principal curvatures* κ_+, κ_-.
These directions are called principal directions

Remark *As it was noted above normal unit vector as well as a shape operator are defined up to a sign. Hence principal curvatures, i.e. eigenvalues of shape operator are defined up to a sign too:*

$$\mathbf{n} \to -\mathbf{n}, \text{then } S \to -S, \;\; \text{then } (\kappa_1, \kappa_2) \to (-\kappa_1, -\kappa_2) \tag{4.11}$$

Remark. *Principal directions are well-defined in the case if principal curvatures (eigenvalues of shape operator) are different:* $\lambda_1 = \kappa_1 \neq \kappa_2 = \lambda_2$. *In the case if eigenvalues* $\lambda_1 = \lambda_2 = \lambda$ *then* $S = \lambda E$ *is proportional to unity operator. In this case all vectors are eigenvectors, i.e. all directions are principal directions. (This happens for the shape operator of the sphere: see the Homework 9.)*

Remark *Does shape operator have always two eigenvectors? Yes, this follows from the fact that shape operator is symmetrical operator. One can prove that*

$$\langle S\mathbf{a}, \mathbf{b}\rangle = \langle S\mathbf{b}, \mathbf{a}\rangle\,,$$

for arbitrary two tangent vectors $\mathbf{a}, \mathbf{b}$,

This implies that principal directions are orthogonal to each other. Indeed one can see that $\lambda_2(\lambda_2, \lambda_1) = (S\lambda_2, \lambda_1) = (\lambda_2, S\lambda_1) = \lambda_1(\lambda_2, \lambda_1)$. *It follows from this relation that eigenvectors are orthogonal* ($(\lambda_-, \lambda_+) = 0$) *if* $\lambda_- \neq \lambda_+$ *If* $\lambda_- = \lambda_+$ *then all vectors are eigenvectors. One can choose in this case* λ_-, λ_+ *to be orthogonal.*

It has to be mentioned that equation (4.9) may be used to prove and to find eigenvectors. Indeed following equation (4.9) consider on unit cirle $|\mathbf{v}| = 1$ *a function*

$$f(\mathbf{v}) = (S(\mathbf{v}), \mathbf{v})$$

One can see that minimu and maximum values of this function define two eigenvalues of oeprator S, *and the points where these extrema atteint define eigenvectors.*

Definition

Gaussian curvature K of the surface M at a point $\mathbf{p}$ is equal to the product of principal curvatures.

$$K = \kappa_1 \kappa_2 \tag{4.12}$$

Recall that the product of eigenvalues of a linear operator is determinant of this operator, Thus we immediately come to the useful formulae for calculating the Gaussian curvature

Proposition *Let S be a shape operator at the point $\mathbf{p}$ on the surface M. Then*

Gaussian curvature K of the surface M at the point $\mathbf{p}$ is equal to the determinant of the shape operator:

$$K = \kappa_1 \kappa_2 = \det S \tag{4.13}$$

E.g. if in a given coordinate basis a shape operator is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (see e.g. equations (4.5) and(4.8)), then

$$K = \det S = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc, \quad H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d \tag{4.14}$$

One can define also so called mean curvature H of the surface. Mean curvature K of the surface M at every point is equal to the sum of the principal curvatures: $H = \kappa_1 + \kappa_2$. Mean curvature H of the surface M at an arbitrary point $\mathbf{p}$ is equal to the trace of the shape operator S at this point: $H = \kappa_1 + \kappa_2 = \operatorname{Tr} S$.

4.1.4 Examples of calculation of Weingarten operator, curvatures for cylinder sphere and for saddle.

Cylinder

We already calculated induced Riemannian metric on the cylinder (see (1.58)).

Cylinder is given by the equation $x^2 + y^2 = a^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}, \quad \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \tag{4.15}$$

Normal unit vector $\mathbf{n} = \pm \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. *Choose* $\mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. *Weingarten operator*

$$S\partial_h = -\partial_{\mathbf{r}_h}\mathbf{n} = -\partial_h \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0\,,$$

$$S\partial_\varphi = -\partial_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\partial_\varphi}{a}.$$

$$S\begin{pmatrix} \partial_h \\ \partial_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-\partial_\varphi}{R} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{R} \end{pmatrix}. \tag{4.16}$$

Principal curvaures are $\kappa_1 = 0$ *and* $\kappa_2 = -\frac{1}{R}$. *For the Gaussian curvature we have*

$$K = \det S = \frac{\det A}{\det G} = \det \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = 0\,. \tag{4.17}$$

If we change $\mathbf{n} \to -\mathbf{n}$ *Gaussian curvature will not change.*

Sphere

Sphere is given by the equation $x^2 + y^2 + z^2 = R^2$.

To calculate Gaussain curvature for arbitrary surface we may choose arbitrary parameterisation, since curvature does dpend on the choice of parameterisation. Usually we choose the parameterisation which is convenient for caluclations. In the case of sphere we have freedom to use an arbitrary parameterisation. We will do calculations for arbitrary parametersiation. (At the end we will do calculations in spherical coordinates just to double check that everything is alright).

Let $\mathbf{r} = \mathbf{r}(u, v)$ *belong to the sphere* $x^2 + y^2 + z^2 = R^2$ *of radius* R*; and let* (u, v) *are arbitrary local coordinates on this sphere*

Consider at the points of the sphere, a unit vector field

$$\mathbf{n}(u, v) = \frac{\mathbf{r}(u, v)}{R}$$

since the surface is sphere hence this unit vector field is orthogonal to the surface of the sphere. Then calculate the shape operator:

$$S(\mathbf{r}_u) = -\partial_{\mathbf{r}_u}\mathbf{n}(u, v) = -\frac{\partial}{\partial u}\left(\frac{\mathbf{r}(u, v}{R)}\right) = -\frac{\mathbf{r}_u}{R}\,,$$

and the same is for the vector $\mathbf{r}_v$*:*

$$S(\mathbf{r}_v) = -\partial_{\mathbf{r}_v}\mathbf{n}(u,v) = -\frac{\partial}{\partial v}\left(\frac{\mathbf{r}(u,v}{R)}\right) = -\frac{\mathbf{r}_v}{R},$$

We see that both vectors are eigenvetors with the same eigenvalue $\frac{-1}{R}$*, i.e.* all *tangent vectors are eigenvectors with teh same eigenvalue. The matrix of shape operator is* $\begin{pmatrix} -\frac{1}{R} & \\ 0 & -\frac{1}{R} \end{pmatrix}$*, the Gaussian curvater is equal to* $K = \frac{1}{R^2}$*.*

We can repeat calculations in specific coordinates, e.g. in spherical coordinates.

Consider the parameterisation of sphere in spherical coordinates

$$\mathbf{r}(\theta,\varphi)\colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{4.18}$$

For the sphere $\mathbf{r}(\theta,\varphi)$ *is orthogonal to the surface. Hence normal unit vector* $\mathbf{n}(\theta,\varphi) = \pm\frac{\mathbf{r}(\theta,\varphi)}{R} = \pm\begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}$*. Choose* $\mathbf{n} = \frac{\mathbf{r}}{R} = \begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}$.
Weingarten operator

$$S\partial_\theta = -\nabla^{\text{can.flat}}_{\mathbf{r}_\theta}\mathbf{n} = -\partial_\theta\mathbf{n} = -\partial_\theta\left(\frac{\mathbf{r}}{R}\right) = -\frac{\mathbf{r}_\theta}{R},$$

$$S\partial_\varphi = -\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi\mathbf{n} = -\partial_\varphi\left(\frac{\mathbf{r}}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}.$$

$$S\begin{pmatrix}\partial_\theta \\ \partial_\varphi\end{pmatrix} = \begin{pmatrix}-\frac{\partial_\theta}{R} \\ -\frac{\partial_\varphi}{R}\end{pmatrix}, \quad S = -\begin{pmatrix}\frac{1}{R} & 0 \\ 0 & \frac{1}{R}\end{pmatrix}. \tag{4.19}$$

For the Gaussian we have

$$K = \det S = \frac{\det A}{\det G} = \det\begin{pmatrix}-\frac{1}{R} & 0 \\ 0 & -\frac{1}{R}\end{pmatrix} = \frac{1}{R^2}, \tag{4.20}$$

Gaussian curvature will not change if we change $\mathbf{n} \to -\mathbf{n}$.

We see that for the sphere Gaussian curvature is not equal to zero, whilst for cylinder and cone Gaussian curvature equals to zero.

Saddle Consider saddle: $z = kxy$:

$$\begin{cases} x = u \\ y = v \\ z = kxy \end{cases} \tag{4.21}$$

It is ruled surface. For every point of saddle $\mathbf{p}\colon \begin{cases} x = u_0 \\ y = v_0 \\ z = ku_0v_0 \end{cases}$ one can consider two stragit lines which pass thrOugh this point and belong to the saddle:

$$l_1\colon \begin{cases} x = u_0 \\ y = v_0 + t \\ z = ku_0(v_0 + t) \end{cases} , -\infty < t < \infty \quad l_2\colon \begin{cases} x = u \\ y = v_0 \\ z = k(u_o + t)v_0 \end{cases} , -\infty < t < \infty$$

Calculate the shape operator and Gaussian curvature of the saddle at the stationary point $x = y = z = 0$, i.e. $u = v = 0$. One can see that for the saddle (4.21)

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ kv \end{pmatrix} , \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ ku \end{pmatrix} ,$$

and one can choose normal unit vector field $\mathbf{n} = \mathbf{n}(u, v)$ as

$$\mathbf{n}(u, v) = \frac{1}{\sqrt{1 + ku^2 + kv^2}} \begin{pmatrix} kv \\ ku \\ -1 \end{pmatrix} ,$$

Indeed it is easy to see that $(\mathbf{n}.n) = 1$ and $(\mathbf{n}, \mathbf{r}_u) = (\mathbf{n}, \mathbf{r}_v) = 0$, i.e. $\mathbf{n}(u, v)$ is unit vector field whic is orthogonal to tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.

Now notice that for the point $u = v = 0$ calculations for $\frac{\partial \mathbf{n}(u,v)}{\partial u}\big|_{u=v=0}$ and $\frac{\partial \mathbf{n}(u,v)}{\partial v}\big|_{u=v=0}$ which lead to calculation of shape operator are simple:

$$\frac{\partial \mathbf{n}(u, v)}{\partial u}\Big|_{u=v=0} = \frac{\partial \left(\frac{1}{\sqrt{1+ku^2+kv^2}} \begin{pmatrix} kv \\ ku \\ -1 \end{pmatrix} \right)}{\partial u}\Bigg|_{u=v=0} = \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} ,$$

and

$$\frac{\partial \mathbf{n}(u,v)}{\partial v}\Big|_{u=v=0} = \frac{\partial\left(\frac{1}{\sqrt{1+ku^2+kv^2}}\begin{pmatrix} kv \\ ku \\ 1\end{pmatrix}\right)}{\partial v}\Big|_{u=v=0} = \begin{pmatrix} k \\ 0 \\ 0\end{pmatrix},$$

since $\frac{1}{\sqrt{1+k^2u^2+k^2v^2}}\big|_{u=v=0} = 1$ *and*

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{1+k^2u^2+k^2v^2}}\right)\Big|_{u=v=0} = \frac{k^2u}{\sqrt{1+k^2u+k^2v}}\Big|_{u=v=0} = 0\,,$$

and

$$\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{1+k^2u^2+k^2v^2}}\right)\Big|_{u=v=0} = \frac{k^2v}{\sqrt{1+k^2u+k^2v}}\Big|_{u=v=0} = 0\,.$$

Hence for the shape operator at the point $u = v = 0$ we have

$$S(\mathbf{r}_u)\big|_{u=v=0} = -\frac{\partial \mathbf{n}(u,v)}{\partial u}\Big|_{u=v=0} = -\begin{pmatrix} 0 \\ k \\ 0\end{pmatrix} = -k\mathbf{r}_v\big|_{u=v=0}$$

and

$$S(\mathbf{r}_v)\big|_{u=v=0} = -\frac{\partial \mathbf{n}(u,v)}{\partial v}\Big|_{u=v=0} = -\begin{pmatrix} k \\ 0 \\ 0\end{pmatrix} = -k\mathbf{r}_u\big|_{u=v=0}\,.$$

Thus at the origin shape operator is $S = \begin{pmatrix} & -k \\ -k & 0\end{pmatrix}$ *and Gaussian curvature at the otigin is equal to*

$$K = \det S = -k^2\,.$$

Saddle has negative curvature at the origin.

4.1.5 Theorem of parallel transport over closed curve (detailed formulation)

We formulated in subsection 4.1.1 very important theorem about parallel transport of vectors over closed curve.

Now after learning the Gaussian curvature we formulate it again in more detail, focusing attention on the fact that using this Theorem we obtain information about Gaussian curvature using only induced Riemannian metric.

We will deduce Theorema Egregium from this theorem.

We recall here formulation of this Theorem in subsection4.1.1 and will formulate it again in more details.

Let M be a surface in Euclidean space $\mathbf{E}^3$. Consider a closed curve C on M, $M\colon \mathbf{r} = \mathbf{r}(u,v)$, $C\colon \mathbf{r} = \mathbf{r}(u(t,v(t)), 0 \leq t \leq t_1$, $\mathbf{x}(0) = \mathbf{x}(t_1)$. $(u(t),v(t)$ are internal coordinates of the curve C.)

Consider the parallel transport of an arbitrary tangent $\mathbf{X}$ vector along the closed curve C: Recall that we did it in the section 3.1.2 where we already considered parallel transport over curve for arbitrary connection and for Levi-Civita connection for arbitrary Riemannian manifold (see equations (3.3) *and* (3.5)*). Now we will repeat these considerations for this special case.)*

$$\mathbf{X}(t) = \underbrace{X^\alpha(t)\frac{\partial}{\partial u_\alpha}\big|_{u^\alpha(t)}}_{\text{Internal observer}} = \underbrace{X^\alpha(t)\mathbf{r}_\alpha\big|_{\mathbf{r}(u(t),v(t))}}_{\text{External observer}}, \left(\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}\right).$$

$$\mathbf{X}(t)\colon\ \frac{\nabla \mathbf{X}(t)}{dt} = 0,\ 0 \leq t \leq t_1\,,$$

i.e.

$$\frac{dX^\alpha(t)}{dt} + X^\beta(t)\Gamma^\alpha_{\beta\gamma}(u(t))\frac{du^\gamma(t)}{dt} = 0,\ 0 \leq t \leq t_1\,, \qquad \alpha,\beta,\gamma = 1,2\,, \quad (4.22)$$

where ∇ is the connection induced on the surface M by canonical flat connection (see (2.23)*), or (it is the same) the Levi-Civita connection* (2.30) *of the induced Riemannian metric on the surface M and $\Gamma^\alpha_{\beta\gamma}$ its Christoffel symbols:*

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\pi}\left(\frac{\partial g_{\pi\alpha}}{\partial u^\beta} + \frac{\partial g_{\pi\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\pi}\right), \text{where}\, g_{\alpha\beta} = \langle \mathbf{r}_\alpha, \mathbf{r}_\beta\rangle = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} \quad (4.23)$$

are components of induced Riemannian metric $G_M = g_{\alpha\beta}du^\alpha du^\beta/$

Let $\mathbf{r}(0) = \mathbf{p}$ be a starting (and ending) point of the curve C: $\mathbf{r}(0) = \mathbf{r}(t_1) = \mathbf{p}$. The differential equation (4.22) *defines the linear operator*

$$P_C\colon T_\mathbf{p}M \longrightarrow T_\mathbf{p}M \quad (4.24)$$

For any vector $\mathbf{X} \in T_\mathbf{p}M$, its image the vector $R_C\mathbf{X}$ as the solution of the differential equation (4.22) *with initial condition $\mathbf{X}(t)\big|_{t=0} = \mathbf{X}$. (See also section 3.1.2, equations* (3.3) *and* (3.5)*).)*

On the other hand we know that parallel transport is orthogonal operator, it does not change the scalar product of two vectors, and it does not chnage lengths of vectors (see (3.5) *in the subsection 3.1.2):*

$$\langle \mathbf{X}, \mathbf{X} \rangle = \langle P_C \mathbf{X}, P_C \mathbf{X} \rangle \tag{4.25}$$

We see that P_C is an orthogonal operator in the 2*-dimensional vector space $T_{\mathbf{p}}M$. We know that orthogonal operator preserving orientation is the operator of rotation on some angle ϕ.*

One can see that P_C preserves orientation [12] *then the action of operator P_C on vectors is rotation on the angle, i.e. the result of parallel transport along closed curve is rotation on the $\angle\phi$. This angle depends depends on the curve. The very beautiful question arises: How to calculate this angle* $\Delta\Phi(C)$

Theorem *Let M be a surface in Euclidean space $\mathbf{E}^3$. Let C be a closed curve C on M such that C is a boundary of a compact oriented domain $D \subset M$. Consider the parallel transport of an arbitrary tangent vector along the closed curve C. As a result of parallel transport along this closed curve any tangent vector rotates through the angle*

$$\angle\phi = \angle\left(\mathbf{X}, P_C \mathbf{X}\right) = \int_D K d\sigma \,, \tag{4.26}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g} dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g} dudv$.

Remark *One can show that the angle of rotation does not depend on initial point of the curve.*

Remark
If C is a smooth closed geodesics then it follows from this Theorem and properties of geodesics that rotation angle is equal to $2\pi n$ (where n is integer). Theorem is valid also for piecewise smooth curve. In general if C is piecewise smooth curve, then one can see that rotation angle is equal to $\sum \alpha_i - \pi(n-1)$ where n is number of smooth arcs, and α_i angles between them (see example above in next subsection).

The proof of this Theorem see in Appendices.

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[12] We will consider mainly the case if the closed curve C is a boundary of a compact oriented domain $D \subset M$. In this case one can see by continuity arguments that operator R_C preserves an orientation.

4.1.6 **Gauß** *Theorema Egregium*

Here we wlll formulate and prove very important corollary of the Theorem on parallel transport over closed curve. This is Gauß Theorema Egregium.

We defined Gaussian curvature in terms of Weingarten (shape) operator as a product of principal curvatures. This definition was in terms of External Observer.

The Theorem about transport over closed curve implies the remarkable Corollary:

Corollary *Gauß Egregium Theorema*

Gaussian curvature of the surface can be expressed in terms of induced Riemannian metric. It is invariant of isometries.

Indeed let D be a small domain around a given point $\mathbf{p}$, let C its boundary and $\angle\phi(D)$ be an angle of rotation. Denote by $S(D)$ an area of this domain. Applying the Theorem for the case when area of the domain D tends to zero we we come to the statement that

$$\textit{if } S(D) \to 0 \textit{ then } \angle\phi(D) = \int_D K d\sigma \to K(\mathbf{p})S(D), \text{ i.e.}$$

$$K(\mathbf{p}) = \lim_{S(D)\to 0} \frac{\angle\phi(D)}{S(D)}\,. \tag{4.27}$$

Now notice that left hand side od this equation defining Gaussian curvature $K(\mathbf{p})$ depends only on Riemannian metric on the surface C. Indeed numerator of LHS is defined by the solution of differential equation (4.22) *which depends on Levi-Civita connection depending on the induced Riemannian metric, and denominator is an area depending on Riemannian metric too. Thus we come to Gauß, Theorema Egregium.*

Note that mean curvature ($H = \kappa_+ + \kappa_- = \operatorname{Tr} S$) is not the invariant of isometries. It cannot be calculated in terms of induced Riemannian metric. E.g. mean curvature of cylindre is equal to $H = 0 + \frac{1}{R} = \frac{1}{R}$. On the other hand cylindre is locally Euclidean, hence mean curvature cannot be expressed in terms of induceed Riemannian metric.

Example

Consider "triangle" ABC on the surface M in $\mathbf{E}^3$, such that the sides, edges of triangle are arcs of great circles, the shortest curves. Here we consider the case if M is a sphere of radius R, but all our considerations and the final formula (4.28) *is valid for arbitrary embedded surface.*

Let α, β, γ be angles between edges of the triangle ABC.

We will apply Theorem on closed curve to this triangle.

One can easy to caclulate the parallel transport of arbitrary vector along geodesic: during parallell transport along geodesics vector remains tangent, if it is tangent at the initial point, and in the case if it is not tangent, the angle between this vector and the tangent vector is preserved. This means that we can calculate for the triangle ABC the left hand side of the formula (4.26).

Denote by $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *a (non-zero) vectors such that they are attached to vertices of this triangle: $A \in T_A M$ $B \in T_B M$ $\mathbf{C} \in T_C M$ and they are tangent to the corresponding sides: vector* $\mathbf{A}$ *is tangent to the side AB, vector* $\mathbf{B}$ *is tangent to the side BC, and vector* $\mathbf{C}$ *is tangent to the side CA. (we suppose that they are oriented anti-clock wise)*

Suppose also that all these vectors have the same length.

Let $\mathbf{X}(t)$ *be a parallel transport of the vector* $\mathbf{X}$ *along the edges of triangle ABC: initial condition is that vector* $\mathbf{X}$ *is attached at the vertex A, and coincides with vector* $\mathbf{A}$*:* $\mathbf{X}(t)|_{t_A} = \mathbf{A}$.

One can see that due to parallel transport vector $\mathbf{X} = \mathbf{A}$ *will rotate on the anlge $\alpha + \beta + \gamma - \pi$*

Show it.

Due to the porperty of parallel trasport along geodesic for parallel transport $\mathbf{X}(t_B)$ *over edges of triangle we see that*

- *For all t: $t_A \leq t < t_B$ the vector* $\mathbf{X}(t)$ *is tangent to the curve* $\mathbf{r}(t)$*, its length is the same and at the point at the final point of the trip it will have the angle $\pi - \beta$. with the vector* $\mathbf{B}$.
- *For all t: $t_B \leq t < t_C$ the vector* $\mathbf{X}(t)$ *will have the angle $\pi - \beta$ with tangent vector to the edge BC. its length is the same and at the final point of the trip it will have the angle $\pi - \gamma + \pi - \beta$. with the vector* $\mathbf{C}$.
- *For all t: $t_B \leq t < t_C$ the vector* $\mathbf{X}(t)$ *will have the angle $2\pi - \beta - \gamma$ with tangent vector to the edge CA. Its length is the same and at the final point of the trip it will have the angle $\pi - \alpha + \pi - \gamma + \pi - \beta$. with the vector* $\mathbf{C}$.

The angle between the final vector and initial will be $3\pi - \alpha - \beta - \gamma$, this meant that final vector $\mathbf{A}$ *will rotate on the angle $-(3\pi - \alpha - \beta - \gamma) = \alpha + \mathbf{b} + \gamma - \pi + 2\pi$.*

Now applu formula (4.26)*: We have that rotation of vector during parallel transport along the boundary of triangle is equal to*

$$\alpha + \beta + \gamma - \pi = \int_{\triangle ABC} K d\sigma \tag{4.28}$$

n particular for sphere of radius R we come to

$$\alpha+\beta+\gamma-\pi=\int_{\triangle ABC} K d\sigma=\frac{\text{the area of }\triangle ABC}{R^2}\,. \tag{4.29}$$

In the case of zero curvatre we come to the formula which everybody knows: sum of angles of triangle is equal to π.

In Appendices we develop the technique which itself is very interesting. One of the applications of this technique is the proof of the Theorem (4.26).

There are different other proofs of Theorema Egregium. See Appendices.

5 Curvature tensor

5.1 Curvature tensor for connection

Definition-Proposition *Let manifold M be equipped with connection ∇. Consider the following operation which assigns to arbitrary vector fields $\mathbf{X},\mathbf{Y}$ and $\mathbf{Z}$ on M the new vector field:*

$$\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}=\left(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}-\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}-\nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z} \tag{5.1}$$

This operation is obviously linear over the scalar coefficients.

One can show that this operation is $C^\infty(M)$-linear with respect to vector fields $\mathbf{X},\mathbf{Y}\,\mathbf{Z}$, i.e. for an arbitrary functions f,g,h

$$\mathcal{R}(f\mathbf{X},g\mathbf{Y})(h\mathbf{Z})=fgh\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\,. \tag{5.2}$$

This means that the operation defines the tensor field of the type $\begin{pmatrix}1\\3\end{pmatrix}$: If $\mathbf{X}=X^i\partial_i,\mathbf{X}=X^i\partial_i,\mathbf{X}=X^i\partial_i$ then according to (5.2)

$$\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}=\mathcal{R}(X^m\partial_m,Y^n\partial_n)(Z^r\partial_r)=Z^rR^i_{rmn}X^mY^n$$

where we denote by R^i_{rmn} the components of the tensor $\mathcal{R}$ in the coordinate basis ∂_i

$$R^i{}_{rmn}\partial_i=\mathcal{R}(\partial_m,\partial_n)\partial_r \tag{5.3}$$

This tensor is called curvature tensor of the connection ∇.

Express components of the curvature tensor in terms of Christoffel symbols of the connection. If $\nabla_m\partial_n=\Gamma^r_{mn}\partial_r$ then according to the (5.1) we have:

$$R^i{}_{rmn}\partial_i=\mathcal{R}(\partial_m,\partial_n)\partial_r=\nabla_{\partial_m}\nabla_{\partial_n}\partial_r-\nabla_{\partial_n}\nabla_{\partial_m}\partial_r,$$

$$
\begin{aligned}
R^i{}_{rmn} &= \nabla_{\partial_m}\left(\Gamma^p_{nr}\partial_p\right) - \nabla_{\partial_n}\left(\Gamma^p_{mr}\partial_p\right) = \\
&\partial_m\Gamma^i_{nr} + \Gamma^i_{mp}\Gamma^p_{nr} - \partial_n\Gamma^i_{mr} - \Gamma^i_{np}\Gamma^p_{mr}\,. \qquad (5.4)
\end{aligned}
$$

The proof of the property (5.2) can be given just by straightforward calculations: Consider e.g. the case $f = g = 1$, then

$$
\begin{gathered}
\mathcal{R}(\mathbf{X},\mathbf{Y})(h\mathbf{Z}) = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}(h\mathbf{Z}) - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}(h\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}(h\mathbf{Z}) = \\
\nabla_{\mathbf{X}}\left(\partial_{\mathbf{Y}}h\mathbf{Z} + h\nabla_{\mathbf{Y}}\mathbf{Z}\right) - \nabla_{\mathbf{Y}}\left(\partial_{\mathbf{X}}h\mathbf{Z} + h\nabla_{\mathbf{X}}\mathbf{Z}\right) - \partial_{[\mathbf{X},\mathbf{Y}]}h\mathbf{Z} - h\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} = \\
\partial_{\mathbf{X}}\partial_{\mathbf{Y}}h\mathbf{Z} + \partial_{\mathbf{Y}}h\nabla_{\mathbf{X}}\mathbf{Z} + \partial_{\mathbf{X}}h\nabla_{\mathbf{Y}}\mathbf{Z} + h\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \\
\partial_{\mathbf{Y}}\partial_{\mathbf{X}}h\mathbf{Z} - \partial_{\mathbf{X}}h\nabla_{\mathbf{Y}}\mathbf{Z} - \partial_{\mathbf{Y}}h\nabla_{\mathbf{X}}\mathbf{Z} + h\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \\
\partial_{[\mathbf{X},\mathbf{Y}]}h\mathbf{Z} - h\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} = \\
h\left[\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}\right] + \left[\partial_{\mathbf{X}}\partial_{\mathbf{Y}}h - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}h - \partial_{[\mathbf{X},\mathbf{Y}]h}\right]\mathbf{Z} = \\
h\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} = h\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\,,
\end{gathered}
$$

since $\partial_{\mathbf{X}}\partial_{\mathbf{Y}}h - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}h - \partial_{[\mathbf{X},\mathbf{Y}]}h = 0$.

5.1.1 Properties of curvature tensor

Tensor $R^i{}_{kmn}$ is expressed through derivatives of Christoffel symbols. In spite this fact it is is much more "pleasant" object than Christoffel symbols, since the latter is not the tensor.

It follows from the definition that the tensor R^i_{kmn} is antisymmetrical with respect to indices m, n:

$$
R^i_{kmn} = -R^i_{knm}\,. \tag{5.5}
$$

One can prove that for symmetric connection this tensor obeys the following identities:

$$
R^i{}_{kmn} + R^i_{mnk} + R^i_{nkm} = 0\,, \tag{5.6}
$$

The curvature tensor corresponding to Levi-Civita connection obeys also another identities too (see the next subsection.)

We know well that If Christoffel symbols vanish in a vicinity of a given point $\mathbf{p}$ in some chosen coordinate system then in general Christoffel symbols do not vanish in arbitrary coordinate systems. E.g. Christoffel symbols of canonical flat connection in $\mathbf{E}^2$ vanish in Cartesian coordinates but do not vanish in polar coordinates. This unpleasant property of Christoffel symbols is due to the fact that Christoffel symbols do not form a tensor.

In particular if a tensor vanishes in some coordinate system, then it vanishes in arbitrary coordinate system too. This implies very simple but important

Proposition *If curvature tensor $R^i{}_{kmn}$ vanishes in some coordinate system, then it vanishes in arbitrary coordinate systems.*

We see that if Christoffel symbols vanish in a vicinity of a given point **p** *in some chosen coordinate system then its Riemannian curvature tensor vanishes in a vicinity of the point* **p** *(see the formula (5.4)) and hence it vanishes locally (in a vicinity of point* **p***) in arbitrary coordinate system.*

In fact one can prove

Theorem *If a connection is symmetric then curvature tensor vanishes in a vicinity of a point if and only there exist local Cartesian coordiantes,n a vicinity of this point i.e. coordinates in which Christoffel symbol of connection vanish.*

5.2 Riemann curvature tensor of Riemannian manifolds.

Let M be Riemannian manifold equipped with Riemannian metric G

In this section we will consider curvature tensor of Levi-Civita connection ∇ of Riemannian metric G.

The curvature tensor for Levi-Civita connection will be called later Riemann curvature tensor, or Riemann tensor.

Using Riemannian metric one can consider Riemann tensor with all low indices

$$R_{ikmn} = g_{ij} R^j{}_{kmn} \tag{5.7}$$

In the subsection above we formulated very important Theorem that vanishing of curvature tensor means that connection is locally flat. For Riemann tensor one can formulate the analogous Theorem. If Riemannina manifold is locally Euclidean, i.e. there exist coordinates $(x^1, \dots, x^n)$ such that $G = (dx^1)^2 + \cdots + (dx^n)^2$ then it is evident that Christoffel symbols of Levi-Civita connection vanish in these coordinates, hence curvature tensor vanishes also. The converse implication is true also:

Theorem *Riemann curvature tensor vanishes if and only if Riemannian manifold is locally Euclidean, i.e. if $R^i_{kmn} \equiv 0$ in a vicinity of the point* **p** *of Riemannian manifold, then in a vicinity of this point there exist local* *coordinates $(x^1, \dots, x^n)$ such that Riemannian metric $G = (dx^1)^2 + \cdots + (dx^n)^2$.*

For Riemann tensor one can consider Ricci tensor,

$$R_{mn} = R^i_{min} \tag{5.8}$$

which is symmetrical tensor: $R_{mn} = R_{nm}$.

One can consider scalar curvature:

$$R = R^i_{kin} g^{kn} = g^{kn} R_{kn} \tag{5.9}$$

where g^{kn} is Riemannian metric with indices above (the matrix $||g^{ik}||$ is inverse to the matrix $||g_{il}||$).

Ricci tensor and scalar curvature also play essential role for formulation of famous Einstein gravity equations. In particular the space without matter the Einstein equations have the following form:

$$R_{ik} - \frac{1}{2} R g_{ik} = 0\,. \tag{5.10}$$

Due to identities (5.5) and (5.6) for curvature tensor Riemann tensor obeys the following identities:

$$R_{ikmn} = -R_{iknm} \quad , \quad R_{ikmn} + R_{imnk} + R_{inkm} = 0 \tag{5.11}$$

Riemann curvature tensor which is curvature tensor for Levi-Civita connection obeys also the following identities:

$$R_{ikmn} = -R_{kimn} \quad , \quad R_{ikmn} = R_{mnki}\,. \tag{5.12}$$

These condition lead to the fact that for 2-dimensional Riemannian manifold the Riemann curvature tensor of Levi-Civita connection has essentially only one non-vanishing component: all components vanish or equal to component R_{1212}up to a sign.

Indeed consider for 2-dimensional Riemannian manifold Riemann tensor R_{ikmn}, where *$i, k, m, n = 1, 2$. Since antisymmetricity with respect to third and fourth indices ($R_{ikmn} = -R_{iknm}$), $R_{ik11} = R_{ik22} = 0$* and *$R_{ik12} = -R_{ik21}$. The same for first and second indices: since antisymmetricity with respect to the the first and second indices ($R_{12mn} = -R_{21mn}$), $R_{11mn} = R_{22mn} = 0$* and *$R_{12mn} = -R_{ik21}$. If we denote $R_{1212} = a$ then*

$$R_{1212} = R_{2121} = a, R_{1221} = R_{2112} = -a \tag{5.13}$$

and all other components vanish.

5.2.1 Relation between Gaussian curvature and Riemann curvature tensor and *Theorema Egregium*

For surfaces in $\mathbf{E}^3$ *Gaussian curvature is equal to half of scalar curvature:*

$$K = \frac{R}{2}\,, \tag{5.14}$$

where $R = R^i_{kin} g^{kn}$ *is scalar curvature of Riemann curvature tensor.*

This equation is the fundamental relation which claims that the Gaussian curvature (the magnitude defined in terms of External observer) equals to the scalar curvature (up to a coefdficient), the magnitude defined in terms of Internal Observer. This gives us another proof of Theorema Egregium. (see the proof of equation (5.14) *in subsection "Theorema Egregium again")*

Consider this little bit more in details.

Let M *be a surface in* $\mathbf{E}^3$ *and* $R^i{}_{kmp}$ *be Riemann tensor, Riemann curvature tensor of Levi-Civita connection. Recall that this means that* $R^i{}_{kmp}$ *is curvature tensor of the connection* ∇*, which is Levi-Civita connection of the Riemannian metric* $g_{\alpha\beta}$ *induced on the surface* M *by standard Euclidean metric* $dx^2 + dy^2 + dz^2$*. Recall that Riemann curvature tensor is expressed via Christoffel symbols of connection by the formula*

$$R^i{}_{kmn} = \partial_m \Gamma^i{}_{nk} + \Gamma^i{}_{mp}\Gamma^p{}_{nk} - \partial_n \Gamma^i{}_{mk} - \Gamma^i{}_{np}\Gamma^p{}_{mk} \tag{5.15}$$

(see he formula (5.4)*) where Christoffel symbols of Levi-Civita connection are defined by the formula*

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right) \tag{5.16}$$

(see Levi-Civita Theorem)

Recall that scalar curvature R *of Riemann tensor equals to* $R = R^i_{kim} g^{km}$*, where* g^{km} *is Riemannian metric tensor with upper indices (matrix* $||g^{ik}||$ *is inverse to the matrix* $||g_{ik}||$*).*

Now consider 2*-dimensional case. One can show that in this case scalar curvature* R *can be expressed via the component* $R_{1212} = a$ *by the formula*

$$R = \frac{2R_{1212}}{\det g} \tag{5.17}$$

where $\det g = \det g_{ik} = g_{11}g_{22} - g_{12}^2$.

To see it note that as it was mentioned in the subsection above the formula for scalar curvature becomes very simple in two-dimensional case (see formulae (5.11) and (5.13) above) and in this case it is very easy to calculate Ricci tensor and scalar curvature R. Indeed let $R_{1212} = a$. For 2-dimensional Riemannian surface all other components of Riemann tensor equal to zero or equal to $\pm a$ (see (5.11) and (5.13)). Show it. Using identities (5.11) and (5.11) we see that

$$R_{11} = R^i{}_{1i1} = R^2{}_{121} = g^{22}R_{2121} + g^{21}R_{1121} = g^{22}R_{1212} = g^{22}a \tag{5.18}$$

$$R_{22} = R^i{}_{2i2} = R^1{}_{212} = g^{11}R_{1212} + g^{12}R_{2221} = g^{11}R_{1212} = g^{11}a \tag{5.19}$$

$$R_{12} = R_{21} = R^i{}_{1i2} = R^1{}_{112} = g^{12}R_{2112} = -g^{12}R_{1212} = -g^{12}a \tag{5.20}$$

Thus using the formula for inverse 2×2 matrix we come to the relation

$$R_{ik} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} g^{22}a & -g^{12}a \\ -g^{21}a & g^{11}a \end{pmatrix} = \frac{1}{\det g}\begin{pmatrix} g_{11}a & g_{12}a \\ g_{21}a & g_{11}a \end{pmatrix},$$

i.e. for 2-dimensional Riemannian manifold

$$R_{ik} = \frac{1}{\det g} R_{1212} g_{ik}\,, \tag{5.21}$$

Hence for scalar curvatre of 2-dimension Riemannian manifold

$$R = R^i{}_{kim}g^{km} = R_{km}g^{km} = \frac{1}{\det g}R_{1212}g_{ik}g^{ik} = \frac{2}{\det g}R_{1212}\,. \tag{5.22}$$

Note that the relations (5.21) and (5.22) imply thatt

$$R_{ik} = \frac{1}{2}Rg_{ik}\,. \tag{5.23}$$

One can say that gravity equation for $n = 2$ are trivial. The mathematical meaning of this formula is the following: equation (5.23) means that variation of functional $S = \int R\sqrt{\det g}d\sigma$ vanishes and this is one of corollaries of Gauss-Bonnet Theorem (see later).

Formula (5.17) expresses scalar curvature for surface in terms of non-trivial component R_{1212}. On the other hand Gaussian curvature of 2-dimensional surface is equal to the half of the scalar curvature (see equation (5.14)). Hence we come to

Proposition *For an arbitrary point of the surface M*

$$K = \frac{R}{2} = \frac{R_{1212}}{\det g}\,. \tag{5.24}$$

where $R = R^i_{kim}g^{km}$ is scalar curvature and K is Gaussian curvature.

We know also that for surface M the scalar curvature R is expressed via Riemann curvature tensor by the formula (5.17). *Hence if we know the Gaussian curvature then we know all components of Riemann curvature tensor (since all components vanish or equal to $\pm a$.). This is nothing but Theorema Egregium! Theorema Egregium (see beginning of the section 4) immediately follows from this Proposition which states that Gaussian curvature is equal (up to a coefficient) to scalar curvature whcih is expressed in terms of Riemannian metric.*

One can check equation (5.24) *just by brute force calculating Riemannian metric and Riemanian curvature tensor (see Appendices)*

Here we consider just a simple example.

Example *Let $M = S^2$ be sphere of radius R in $\mathbf{E}^3$. Show that one cannot find local coordinates u, v on the sphere such that induced Riemannian metric equals to $du^2 + dv^2$ in these coordinates.*

This immediately follows from the Proposition. Indeed suppose there exist local coordinates u, v on the sphere such that induced Riemannian metric equals to $du^2 + dv^2$, i.e. Riemannian metric is given by unity matrix. Then according to the formulae for Levi-Civita connection, the Christoffel symbols equal to zero in these coordinates. Hence Riemann curvature tensor equals to zero, and scalar curvature too. Due to Proposition this is in contradiction with the fact that Gaussian curvature of the sphere equals to $\frac{1}{R^2}$.

(The straightforward proof see in the next paragraph)

5.3 Gauss Bonnet Theorem

Consider the integral of curvature over whole closed surface M. According to the Theorem above the answer has to be equal to 0 *(modulo 2π), i.e. $2\pi N$ where N is an integer, because this integral is a limit when we consider very small curve. We come to the formula:*

$$\int_D K d\sigma = 2\pi N$$

(Compare this formula with formula (4.26)*).*

What is the value of integer N?

We present now one remarkable Theorem which answers this question and prove this Theorem using the formula (4.26).

Let M be a closed orientable surface.[13] *All these surfaces can be classified up to a diffeomorphism. Namely arbitrary closed oriented surface M is diffeomorphic either to sphere (zero holes), or torus (one hole), or pretzel (two holes),... "Number k" of holes is intuitively evident characteristic of the surface. It is related with very important characteristic—Euler characteristic $\chi(M)$ by the following formula:*

$$\chi(M) = 2(1 - g(M)), \quad \text{where } g \text{ is number of holes} \tag{5.25}$$

Remark *What we have called here "holes" in a surface is often referred to as "handles" attached o the sphere, so that the sphere itself does not have any handles, the torus has one handle, the pretzel has two handles and so on. The number of handles is also called genus.*

Euler characteristic appears in many different way. The simplest appearance is the following:

Consider on the surface M an arbitrary set of points (vertices) connected with edges (graph on the surface) such that surface is divided on polygons with (curvilinear sides)—plaquets. ("Map of world")

Denote by P number of plaquets (countries of the map)

Denote by E number of edges (boundaries between countries)

Denote by V number of vertices.

Then it turns out that

$$P - E + V = \chi(M) \tag{5.26}$$

It does not depend on the graph, it depends only on how much holes has surface.

E.g. for every graph on M, $P - E + V = 2$ if M is diffeomorphic to sphere. For every graph on M $P - E + V = 0$ if M is diffeomorphic to torus.

Now we formulate Gauß-Bonnet Theorem.

Let M be closed oriented surface in $\mathbf{E}^3$.

Let $K(p)$ be Gaussian curvature at any point p of this surface.

[13] Closed means compact surface without boundaries. Intuitively orientability means that one can define out and inner side of the surface. In terms of normal vectors orientability means that one can define the continuous field of normal vectors at all the points of M. The direction of normal vectors at any point defines outward direction. Orientable surface is called oriented if the direction of normal vector is chosen.

Theorem *(Gauß -Bonnet) The integral of Gaussian curvature over the closed compact oriented surface M is equal to 2π multiplied by the Euler characteristic of the surface M*

$$\frac{1}{2\pi}\int_M K d\sigma = \chi(M) = 2(1 - \textit{number of holes}) \qquad (5.27)$$

In particular for the surface M diffeomorphic to the sphere $\kappa(M) = 2$, for the surface diffeomorphic to the torus it is equal to 0.

The value of the integral does not change under continuous deformations of surface! It is integer number (up to the factor π) which characterises topology of the surface.

E.g. consider surface M which is diffeomorphic to the sphere. If it is sphere of the radius R then curvature is equal to $\frac{1}{R^2}$, area of the sphere is equal to $4\pi R^2$ and left hand side is equal to $\frac{4\pi}{2\pi} = 2$.

If surface M is an arbitrary surface diffeomorphic to M then metrics and curvature depend from point to the point, Gauß -Bonnet states that integral nevertheless remains unchanged.

Very simple but impressive corollary:

Let M be surface diffeomorphic to sphere in $\mathbf{E}^3$. Then there exists at least one point where Gaussian curvature is positive.

Proof: Suppose it is not right. Then $\int_M K\sqrt{\det g}dudv \leq 0$. On the other hand according to the Theorem it is equal to 4π. Contradiction.

Proof of Gauß-Bonet Theorem

Consider triangulation of the surface M. Suppose M is covered by N triangles. Then number of edges will be $3N/over2$. If V number of vertices then according to Euler Theorem

$$N - \frac{3N}{2} + V = V - \frac{N}{2} = \chi(M).$$

Calculate the sum of the angles of all triangles. On the one hand it is equal to $2\pi V$. On the other hand according the formula (4.26) it is equal to

$$\sum_{i=1}^{N}\left(\pi + \int_{\triangle_i} K d\sigma\right) = \pi N + \sum_{i=1}^{N}\left(\int_{\triangle_i} K d\sigma\right) = N\pi + \int_M K d\sigma$$

We see that $2\pi V = N\pi + \int_M K d\sigma$, i.e.

$$\int_M K d\sigma = \pi\left(2V - \frac{N}{2}\right) = 2\pi\chi(M) \blacksquare$$