Gauss Bonnet

$$\frac{1}{2\pi}\int K\,d\sigma = \chi(M)$$

$$\frac{1}{4\pi}\underbrace{\int R\sqrt{g}\,dS}_{\text{action of}} = \chi(M)$$

Euler character,

$\chi(S^2) = 2$

$\chi(T) = 0$

$\chi(\;) = 2(1-p)$

Proof

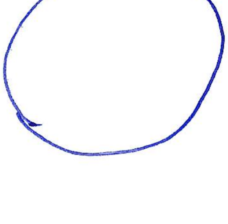
(Physical:

$$S = \int R\sqrt{g}\,dS$$

$$\delta S = \int \Big(\underbrace{R_{ik} - \tfrac{1}{2} R g_{ik}}_{0}\Big)\,\delta g^{ik} = 0.$$

Of course it is not proof.

Examples, sphere $R = 2K = \frac{2}{R^2}$ $\quad \frac{1}{4\pi}\int R\sqrt{g}\,dS = \frac{1}{4\pi}\cdot\frac{2}{R^2}\cdot 4\pi R^2 = 2$



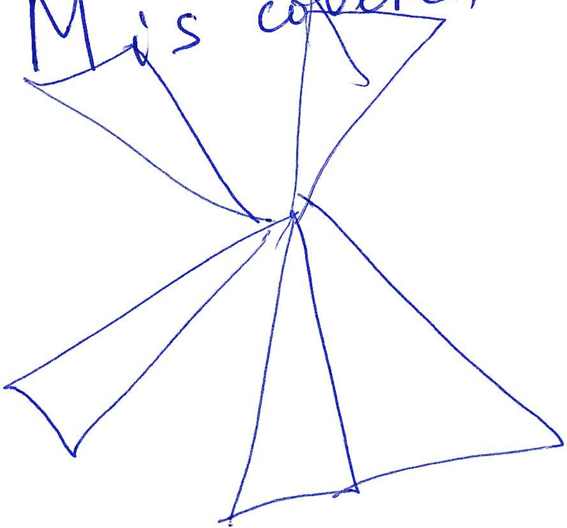
2-example. - Torus

Torus can be embedded in $\mathbb{R}^4$ $\begin{cases} x^2+y^2=1 \\ z^2+t^2=1 \end{cases}$

Torus $= S^1 \times S^1$ $\quad \underline{R=0}$

Consider triangulation of M

M is covered by N triangles.



Number of plaquets - N

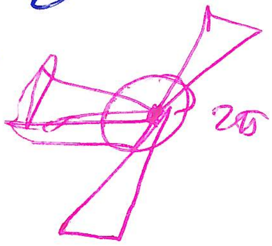
number of Vertices - V

number of edges $\frac{3N}{2}$

$$N - \frac{3N}{2} + V = \chi(M)$$

Calculate sum of angles of triangles

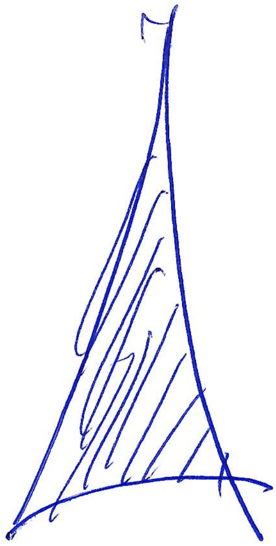
N triangles, V, vertices, $\frac{3N}{2}$ edges



$$\sum(\alpha_i+\beta_i+\gamma_i) = 2\pi V$$

$$\sum(\alpha_i+\beta_i+\gamma_i-\pi) = \int_{\Delta_i} K\,d\sigma$$

$$\Downarrow$$



$$\sum(\alpha_i+\beta_i+\gamma_i) = \pi\cdot N + \int_M K\,d\sigma$$

$$\Downarrow$$

$$2\pi V = \pi N + \int_M K\,d\sigma$$

$$\int_M K\,d\sigma = 2\pi V - \pi N = 2\pi \underbrace{\left(\cancel{2}V - \frac{N}{2}\right)}_{\chi(M)}$$

$$\frac{1}{2\pi}\int K\,d\sigma = \chi(M)$$

$$\chi(M) = \underbrace{N}_{\text{plaquets}} - \underbrace{\frac{3N}{2}}_{\text{edges}} + \underbrace{V}_{\text{vert.}} = V - \frac{N}{2}$$

$$\chi(M) = \sum (-1)^k b_k$$

$$b_k = \dim H^k(M) \quad \text{(Betti number)}$$

$$b_k = \dim Z^k - \dim B^k$$

$$\dim \Lambda_k = \dim \operatorname{Im} d : (\Lambda_k \to \Lambda_{k+1}) + \dim \ker d : (\Lambda_k \to \Lambda_{k+1})$$

$$\dim \Lambda_k = \dim Z_k - \dim B_{k+1}$$

$$\Downarrow$$

$$\sum_{k=1}^{N} (-1)^k \dim \Lambda_k = \sum_{k=1}^{N} (-1)^k \left[\dim Z_k - \dim B_k\right]$$

$$= \chi(M)$$