Tutorial - 1

Exercise

Calculation of $\{\Gamma^i_{km}\}$ for Sphere using 'free particle'

$$G = g_{ik}(x)\dot{x}^i\dot{x}^k \longrightarrow L = \frac{1}{2} g_{ik}(x)\dot{x}^i\dot{x}^k$$

Equation of motion $\Longleftrightarrow$ Geodesics

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}$$

$$\frac{1}{2}\frac{\partial g_{mn}(x)}{\partial x^i}\dot{x}^m\dot{x}^n = \frac{d}{dt}\left[g_{ik}(x)\dot{x}^k\right]$$

$$\dot{x}^m\frac{\partial g_{ik}(x)}{\partial x^m}\dot{x}^k + g_{ik}(x)\ddot{x}^k = \frac{1}{2}\frac{\partial g_{mn}(x)}{\partial x^i}\dot{x}^m\dot{x}^n$$

$$g_{ik}(x)\ddot{x}^k = \frac{1}{2}\left[\frac{\partial g_{mn}(x)}{\partial x^i}\dot{x}^m\dot{x}^n - 2\frac{\partial g_{in}(x)}{\partial x^m}\dot{x}^m\dot{x}^n\right] =$$

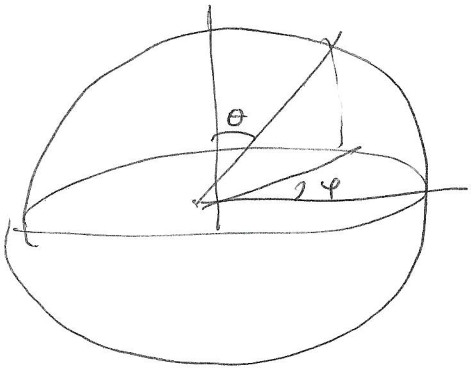
$$= \frac{1}{2}\left[\frac{\partial g_{mn}(x)}{\partial x^i} - \frac{\partial g_{in}(x)}{\partial x^m} - \frac{\partial g_{im}(x)}{\partial x^n}\right]\dot{x}^m\dot{x}^n$$

$$g_{ik}(x)\ddot{x}^k + \frac{1}{2}\left[\frac{\partial g_{im}(x)}{\partial x^n} + \frac{\partial g_{in}(x)}{\partial x^m} - \frac{\partial g_{mn}(x)}{\partial x^i}\right]\dot{x}^m\dot{x}^n = 0$$

$$\ddot{x}^i + \frac{1}{2}g^{ir}\left[\frac{\partial g_{rm}(x)}{\partial x^n} + \frac{\partial g_{rn}(x)}{\partial x^m} - \frac{\partial g_{mn}(x)}{\partial x^i}\right]\dot{x}^m\dot{x}^n$$

$$\underbrace{\ddot{x}^i + \Gamma^i_{mn}\dot{x}^m\dot{x}^n = 0}_{\text{equation of geodesics}}$$

Tutorial - 2



$$dl^2 = R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$L_{free} = \frac{1}{2}R^2(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2)$$

free particle on sphere

$$\frac{\partial L}{\partial \dot\theta} = R^2\dot\theta \qquad \frac{\partial L}{\partial \dot\varphi} = R^2\sin^2\theta\,\dot\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \theta} \qquad R^2\ddot\theta = R^2\sin\theta\cos\theta\,\dot\varphi^2$$

$$\boxed{\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 = 0} \quad (1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = \frac{\partial L}{\partial \varphi} = 0 \qquad \frac{d}{dt}(R^2\sin^2\theta\,\dot\varphi) = 0.$$

$R^2\sin^2\theta\,\dot\varphi = M$

Integral.

$$\sin^2\theta\,\ddot\varphi + 2\sin\theta\cos\theta\,\dot\theta\dot\varphi = 0$$

$$\boxed{\ddot\varphi + 2\frac{\cos\theta}{\sin\theta}\dot\theta\dot\varphi = 0} \quad (2)$$

$$(1) \approx \quad \ddot\theta + \Gamma^\theta_{\varphi\varphi}\dot\varphi\dot\varphi = 0$$

$$\boxed{\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta}$$

$$\boxed{\Gamma^\theta_{\varphi\theta} = 2\cot\theta}$$

$$(2) \approx \quad \ddot\varphi + 2\underbrace{\Gamma^\theta_{\varphi\theta}}_{\text{symmetric}}\dot\theta\dot\varphi = 0$$