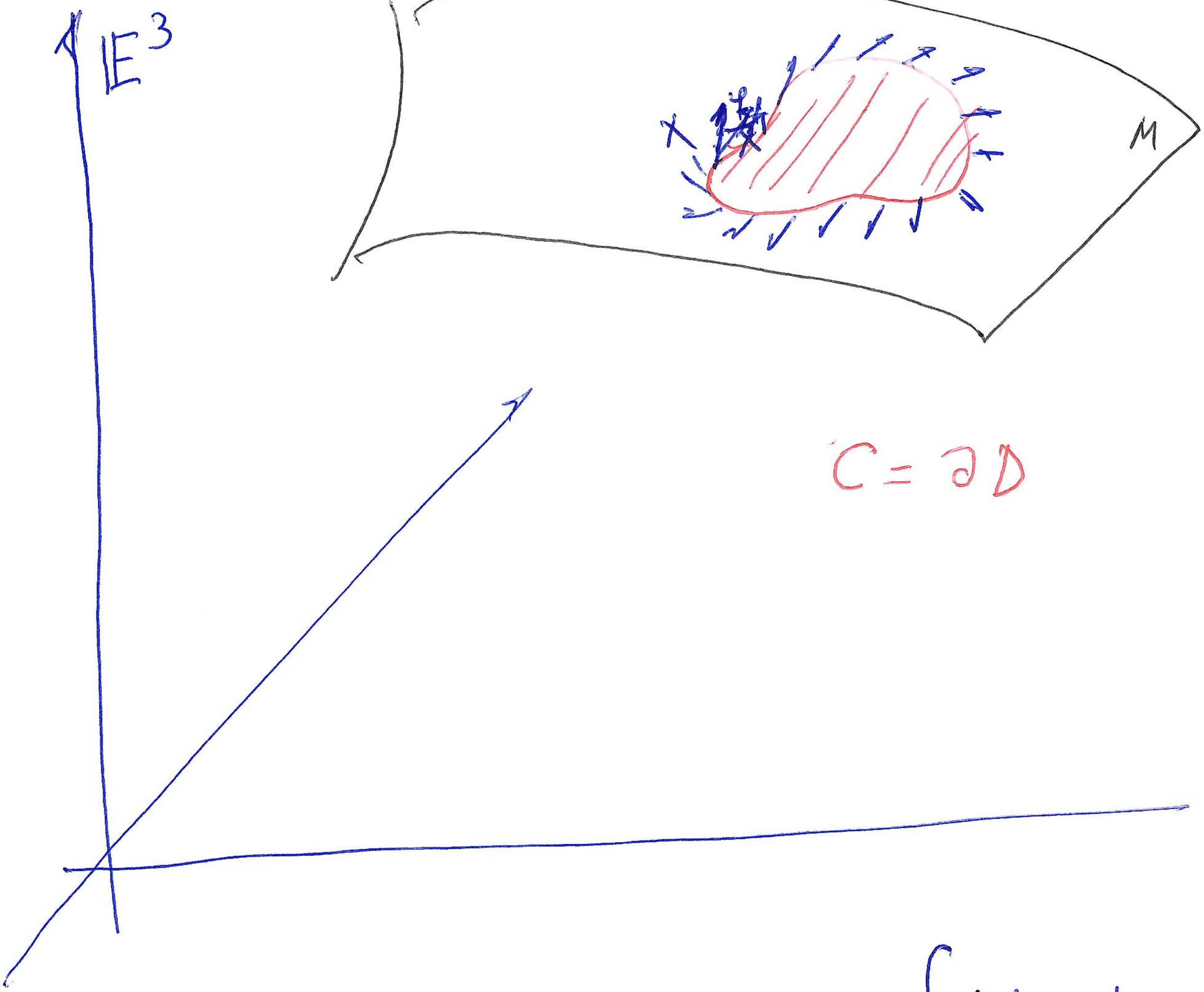
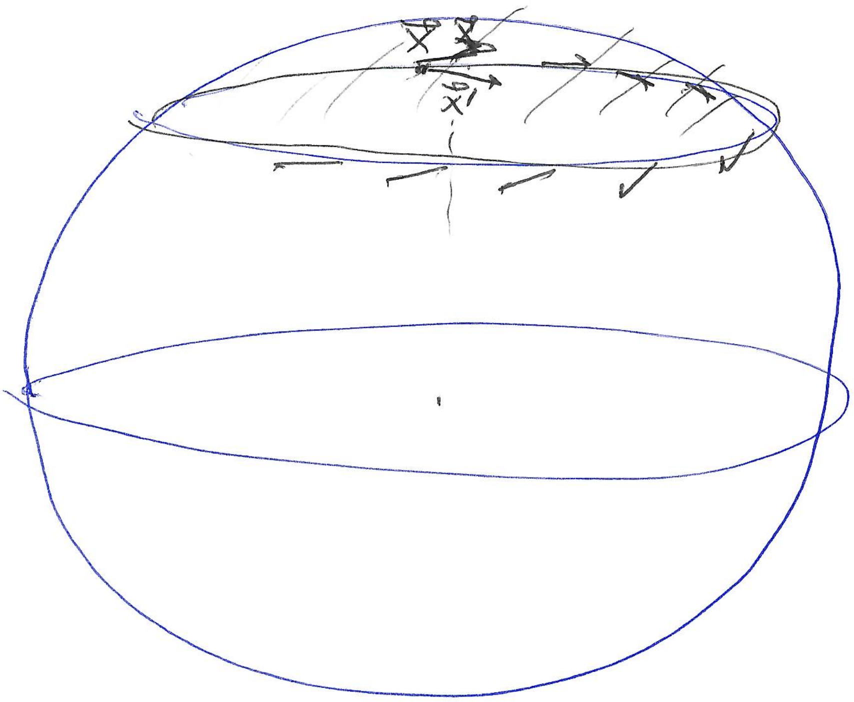


$$\angle \Phi = \angle\left(\vec{X}, P_c \vec{X}\right) = \int_D K \, d\sigma$$

K – Gaussian curvature

$d\sigma = \sqrt{\det g}\, du\, dv$



$$\angle \phi(\vec{X}, P_c \vec{X}) = 2\pi(1-\cos\theta_0)$$

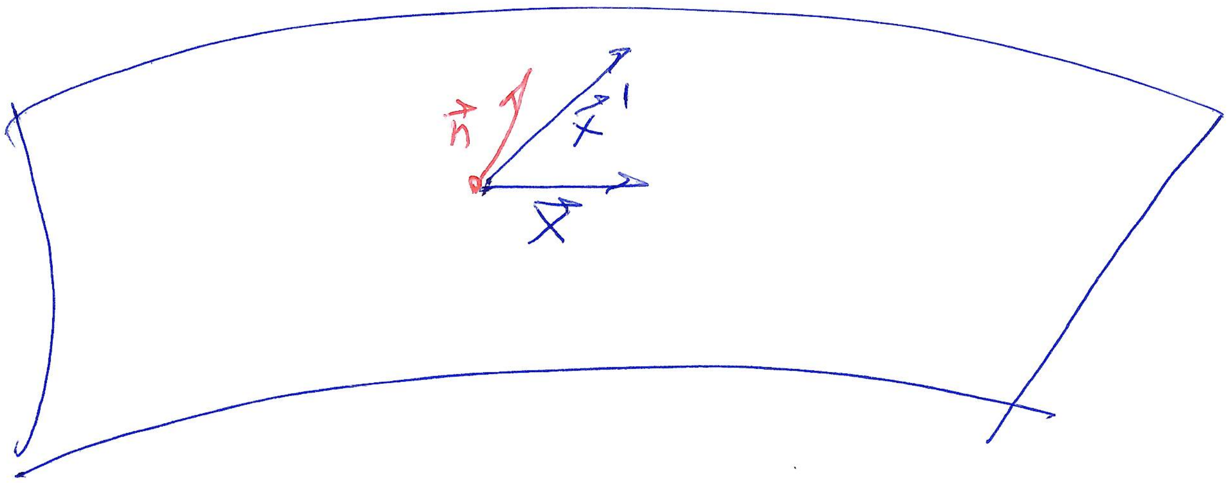
$$C = \partial D \qquad S(D) = 2\pi R H = 2\pi R^2(1-\cos\theta_0)$$

$$K = \frac{1}{R^2}$$ (we will learn it)

$$\angle\phi = \int K \, d\sigma = \oint \frac{1}{R^2}\cdot 2\pi R^2(1-\cos\theta_0)$$

$$\boxed{\angle\phi = 2\pi(1-\cos\theta_0)}$$

Weingarten (Shape) OPERATOR on Surfaces



$\vec{n}(u,v)$ - normal unit vector

$$S: T_p M_0 \longrightarrow T_p M_0$$

$$S(\vec{X}) = -\partial_{\vec{X}} \vec{n}$$

$$\vec{X} = X_u \frac{\partial}{\partial u} + X_v \frac{\partial}{\partial v}$$

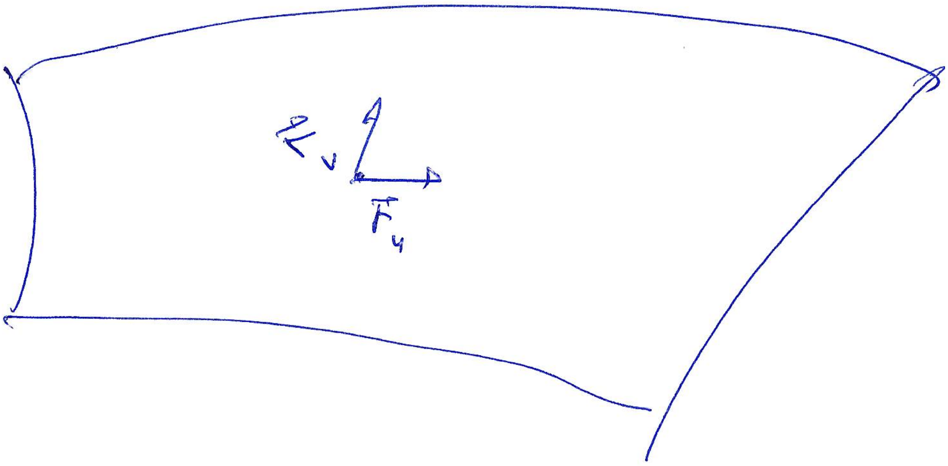
$$S(X_u \partial_u + X_v \partial_v) = -X_u \frac{\partial \vec{n}(u,v)}{\partial u} - X_v \frac{\partial \vec{n}(u,v)}{\partial v}$$

Definition-Proposition $S: T_p M \to T_p M$

Proof $\quad (\vec{n}, \vec{n}) = 1 \Rightarrow$

$$0 = \partial_{\vec{X}} (\vec{n}, \vec{n}) = 2(\partial_{\vec{X}} \vec{n}, \vec{n}) = 2(\vec{X}', \vec{n}) \Rightarrow \vec{X}' \in T_p M.$$

$$K = \det S, \quad H = \operatorname{Tr} S.$$



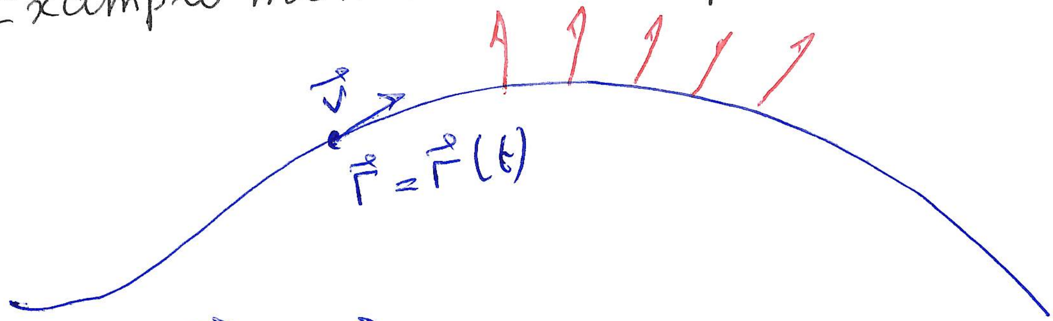
$$S(\vec r_u) = -\frac{\partial \vec n}{\partial u} = a\vec r_u + c\vec r_v$$

$$S(\vec r_v) = -\frac{\partial \vec n}{\partial v}$$

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ in the coordinate basis $\vec r_u, \vec r_v$

$$K = \det S$$

Example motivation Shape operator for curve

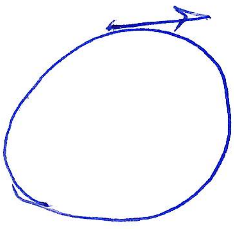


$$\vec X = c\vec v$$

$$S(\vec X) = -\partial_{\vec X}\vec n = -c\,\frac{dx^i(t)}{dt}\,\frac{\partial \vec n(\vec r(t))}{dt} = -c\,\frac{\partial \vec n(\vec r(t))}{\partial t}$$

$$\frac{\partial \vec n(\vec r(t))}{\partial t} = c'\vec v$$

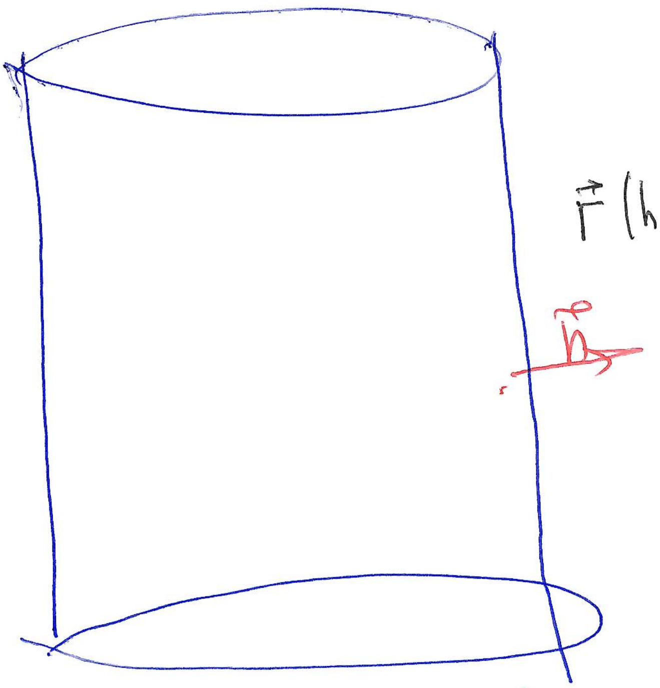
$$S\vec X = \frac{\vec X}{R}$$



$$K = \frac{1}{R}$$

$$K = \left|\frac{d^2\vec r(t)}{dt^2}\right|$$ in natural parametrisation.

Shape operator for cylinder



$$\vec{r}(h,\varphi)\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}$$

$\vec{n} =$

$$\vec{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}$$

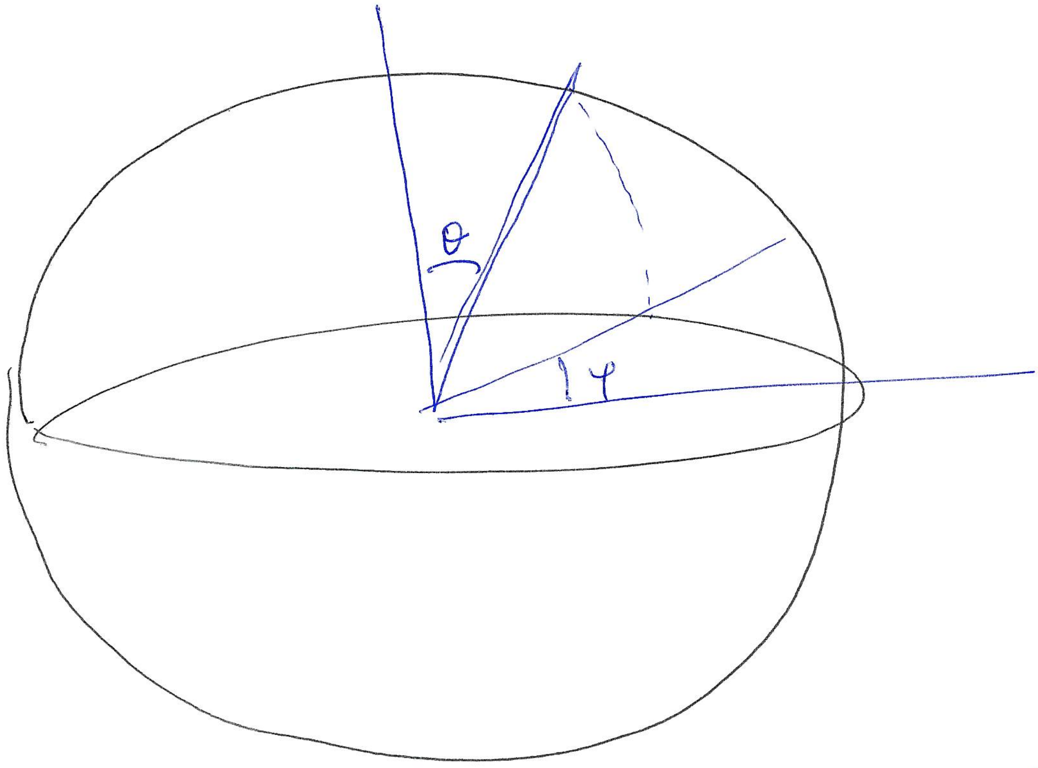
$$\vec{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

$$S\vec{r}_h = -\frac{\partial \vec{n}(\varphi)}{\partial h} = 0$$

$$S\vec{r}_\varphi = -\frac{\partial \vec{n}(\varphi)}{\partial \varphi} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \frac{\vec{r}_\varphi}{R}$$

$$S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$$

$$K = 0; \quad H = \frac{1}{R}$$



$S: \quad x^2 + y^2 + z^2 = 1 \qquad \underline{\vec{r} \cdot \vec{r} = R^2}$

$$\vec{r} = \vec{r}(\theta, \varphi) \qquad \vec{n} = \frac{\vec{r}}{R}$$

$$S(\vec{r}_\theta) = -\partial_\theta \vec{n}(\theta, \varphi) = -\frac{\vec{r}_\theta}{R}$$

$$S(\vec{r}_\varphi) = -\partial_\varphi \vec{n}(\theta, \varphi) = -\frac{\vec{r}_\varphi}{R}$$

$$S = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$$

$$G = \frac{1}{R^2} \qquad \left(H = -\frac{2}{R} \right)$$