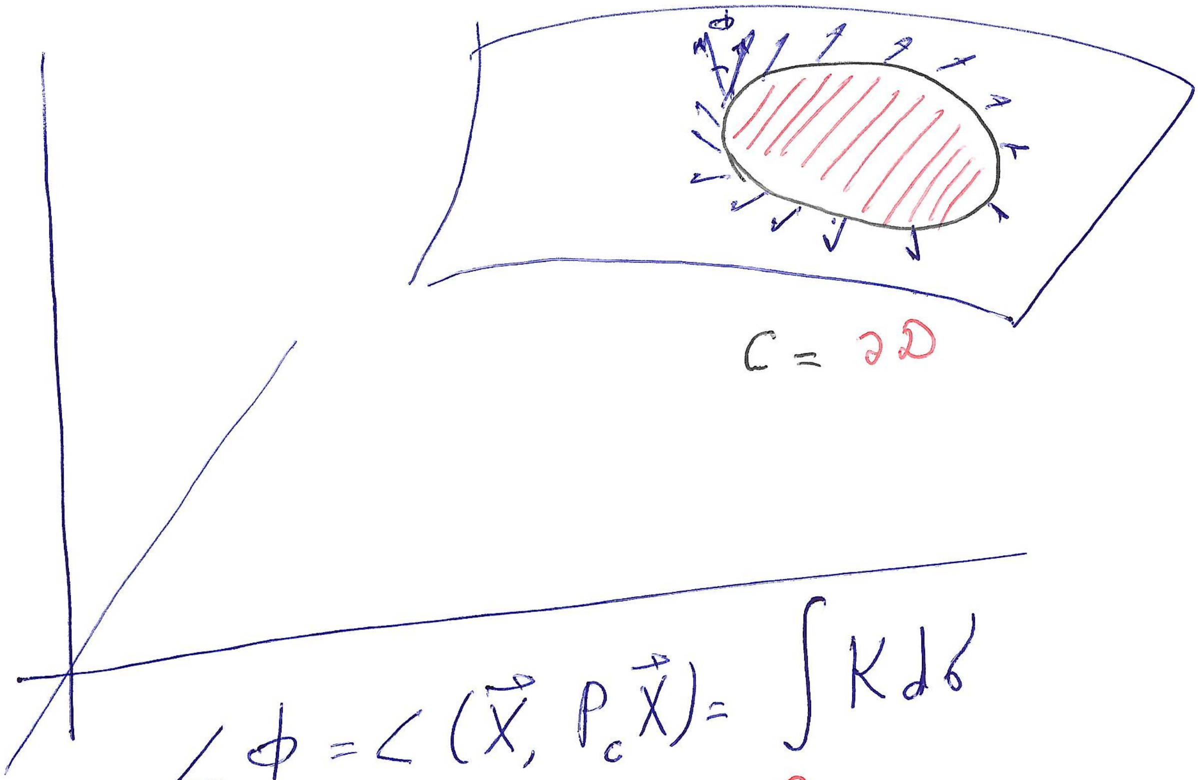
# Theorem of parallel transport over closed curve (detailed formulation)




$$\angle \Phi = \angle(\vec{X}, P_C \vec{X}) = \int_D K \, d\sigma$$

K – Gaussian curvature

$C: \ x^i(t) \quad t_1 \le t \le t_2$

$\underbrace{x^i(t) = \vec{r}(t)}_{\text{external coordinates of curve}} \qquad \vec{r} = \vec{r}(u, v) \qquad \underbrace{(u(t), v(t))}_{\text{internal coordinates}}$

Consider parallel transport

$$\vec{X}(t) = X^1(t)\vec{r}_1(t) + X^2(t)\vec{r}_2(t)$$

external observer

internal observer ['aunt' mathematician] deals with $\underbrace{u, v,}_{\text{coordinates}}$

$$\vec{X} = X^\alpha(t)\frac{\partial}{\partial u^\alpha} \quad \alpha = 1, 2 \quad u^1 = u, \ u^2 = v$$

parallel transport: $\dfrac{dX^\alpha(t)}{dt} + X^\beta(t)\,\Gamma^\alpha_{\beta\gamma}(u(t))\,\dfrac{du^\gamma(t)}{dt} = 0 \qquad (*)$

$$, \ \frac{\nabla \vec{X}(t)}{dt} = 0$$

$\nabla$ is the Levi-Civita connection induced on surface $M$

$$\Gamma^{\gamma}_{\alpha\beta}(u,v) = \frac{1}{2} g^{\gamma\sigma}\left(\frac{\partial g_{\sigma\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\sigma\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\sigma}}\right)$$

$$g_{\alpha\beta} = \langle \vec{r}_{\alpha}, \vec{r}_{\beta} \rangle = \frac{\partial x^i}{\partial u^{\alpha}} \frac{\partial x^i}{\partial u^{\beta}}$$

$$\angle(\vec{X}, P_c \vec{X}) = \int_D K \, d\sigma$$

Aunt mathematician will solve equation (*) and 'he' ('she') will calculate L.H.S. in terms of metric.

External observer will calculate R.H.S. in terms of ambient space.

$$D \to 0$$

$$S(D) \to 0$$

$$\angle \text{ (of rotation)} \to K(p)\, S(D)$$

$$\Downarrow$$

$$K(p) = \lim_{S(D)\to 0} \frac{\angle \text{ (rotation)}}{S(D)}$$

all this can be calculated by aunt in terms of riem. metric.

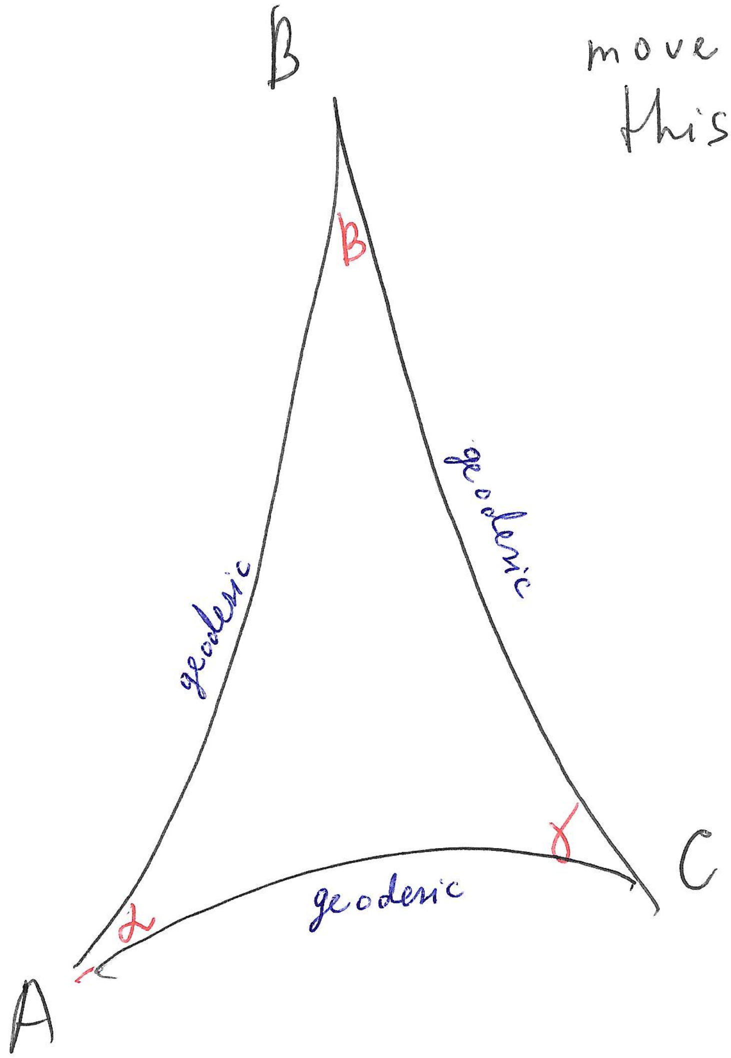
# Gauß Theorema Egregium

Gaussian curvature $K$ can be expressed in terms of induced Riemannian metric.

It is invariant of isometries.

Example

Take tangent vector and move it along sides of this geodesic triangle




$$\alpha + \beta + \gamma - \pi = \int K \, d\sigma$$

for sphere

$$\alpha + \beta + \gamma - \pi = \frac{1}{R^2} \cdot S_{\Delta ABC}$$