$$R(\vec{X},\vec{Y})\vec{Z} =$$

$$= \left(\nabla_{\vec{X}}\nabla_{\vec{Y}} - \nabla_{\vec{Y}}\nabla_{\vec{X}} - \nabla_{[\vec{X},\vec{Y}]}\right)\vec{Z}$$

Surprisingly it is $C^\infty(M)$-linear
(it does not possess derivatives).

$$R(f\vec{X},\, g\vec{Y})(h\vec{Z}) = \quad (*)$$

$$= fgh\, R(\vec{X},\vec{Y})\vec{Z}, \text{ i.e.}$$

$$R(\vec{X},\vec{Y})(\vec{Z}) = R\left(X^{i}(x)\partial_{i},\, Y^{m}(x)\partial_{m}\right)\left(Z^{r}(x)\partial_{r}\right) =$$

$$= X^{i}(x)\,Y^{m}(x)\,Z^{r}(x)\underbrace{\left[R(\partial_{i},\partial_{m})(\partial_{r})\right]}_{\text{vector} = \underbrace{R^{i}{}_{rmn}}_{\text{tensor of connection}}\partial_i}$$

Check (*) for $f = g = 1$

$$\left(\nabla_{\vec{x}}\nabla_{\vec{y}} - \nabla_{\vec{y}}\nabla_{\vec{x}} - \nabla_{[\vec{x},\vec{y}]}\right)(h\vec{z}) =$$

$$= \nabla_{\vec{x}}\left[\left(\partial_{\vec{y}} h\, \vec{z} + h \nabla_{\vec{y}} \vec{z}\right)\right] -$$

$$- \nabla_{\vec{y}}\left[\left(\partial_{\vec{x}} h\, \vec{z} + h \nabla_{\vec{x}} \vec{z}\right)\right] -$$

$$- \partial_{[\vec{x},\vec{y}]} h\, \vec{z} - h \nabla_{[\vec{x},\vec{y}]} \vec{z} =$$

$$= \partial_{\vec{x}}\partial_{\vec{y}} h\, \vec{z} + \cancel{\partial_{\vec{y}} h \nabla_{\vec{x}} \vec{z}} + \cancel{\partial_{\vec{x}} h \nabla_{\vec{y}} \vec{z}} + h \nabla_{\vec{x}} \nabla_{\vec{y}} \vec{z} -$$

$$- \partial_{\vec{y}}\partial_{\vec{x}} h\, \vec{z} - \cancel{\partial_{\vec{x}} h \nabla_{\vec{y}} \vec{z}} - \cancel{\partial_{\vec{y}} h \nabla_{\vec{x}} \vec{z}} - h \nabla_{\vec{y}} \nabla_{\vec{x}} \vec{z} -$$

$$- \partial_{[\vec{x},\vec{y}]} h\, \vec{z} - h \nabla_{[\vec{x},\vec{y}]} \vec{z}$$

$$= \underbrace{\left(\partial_{\vec{x}}\partial_{\vec{y}} h - \partial_{\vec{y}}\partial_{\vec{x}} h - \partial_{[\vec{x},\vec{y}]} h\right)}_{0} \vec{z} +$$

$$+ h\left(\nabla_{\vec{x}}\nabla_{\vec{y}} \vec{z} - \nabla_{\vec{y}}\nabla_{\vec{x}} \vec{z} - \nabla_{[\vec{x},\vec{y}]}\right)\vec{z} =$$

$$= h\left[R(\vec{x},\vec{y})\vec{z}\right]$$

$$R(\partial_m, \partial_n)\partial_r = R^i{}_{rmn}\, \partial_i \qquad (3)$$

$R^i{}_{rmn}$ – $\binom{1}{3}$ tensor

Curvature tensor of connection.

Curvature in terms of Christoffel symbols

$$R^i{}_{rmn}\partial_i = R(\partial_m, \partial_n)\partial_r = \nabla_m \nabla_n \partial_r - \nabla_n \nabla_m \partial_r =$$

$$= \nabla_m (\Gamma^p_{nr}\partial_p) - \nabla_n (\Gamma^p_{mr}\partial_p) =$$

$$= \partial_m \Gamma^p_{nr}\partial_p + \Gamma^p_{nr}\Gamma^i_{mp}\partial_i - \partial_n \Gamma^p_{mr}\partial_p - \Gamma^p_{mr}\Gamma^i_{np}\partial_i$$

$$R^i{}_{rmn} = \left(\partial_m \Gamma^i_{nr} + \Gamma^i_{mp}\Gamma^p_{nr} - \partial_n \Gamma^i_{mr} - \Gamma^i_{np}\Gamma^p_{mr}\right)$$

$R^i{}_{rmn}$ is a tensor!!!

Corollary if R vanishes in some coordinates then it vanishes in arbitrary coordinates [Compare with Christoffel symbols]

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for symmetric connection.

$$\boxed{R \equiv 0} \Longleftrightarrow \text{there exist coordinates: } \Gamma^\alpha_{\beta\gamma} \equiv 0.$$

Gaussian Curvature and $R^{i}{}_{kem.}$

$$\underbrace{R_{ik}}_{\text{Ricci tensor}} = R^{r}{}_{irk}$$

$$\left(\text{Einstein eq. in vacuum} \quad R_{ik} = 0 \right).$$

$$\underbrace{R}_{\text{Scaler}} = g^{ik} R_{ik}$$

$$K = \frac{R}{2} = \frac{R_{1212}}{\det g}$$