Appendices to lectures

Here I put appendices to the lecture notes on Riemannian geometry
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Contents

1 Examples of surfaces in $\mathbf{E}^3$

1.1 Hyperboloids and other quadratic surfaces

One-sheeted and two-sheeted hyperboloids.

These examples were not considered on lectures, but they are interesting for learning purposes.

Consider surface given by the equation

$$x^2 + y^2 - z^2 = c$$

If $c = 0$ it is a cone. We considered it already above.

If $c > 0$ it is one-sheeted hyperboloid—connected surface in $\mathbf{E}^3$.

If $c < 0$ it is two-sheeted hyperboloid— a surface with two sheets: upper sheet $z > 0$ and another sheet: $z < 0$.

Consider these cases separately.

1) *One-sheeted hyperboloid*: $x^2 + y^2 - z^2 = a^2$. It is ruled surface.

Exercise† Find the lines on two-sheeted hyperboloid

One-sheeted hyperboloid is given by the equation $x^2 + y^2 - z^2 = a^2$. It is convenient to choose parameterisation:

$$\mathbf{r}(\theta, \varphi)\colon \quad \begin{cases} x = a\cosh\theta\cos\varphi \\ y = a\cosh\theta\sin\varphi \\ z = a\sinh\theta \end{cases} \tag{1.1}$$

$$x^2 + y^2 - z^2 = a^2\cosh^2\theta - a^2\sinh^2\theta = a^2\,.$$

(Compare the calculations with calculations for sphere! We changed functions $\cos, \sin$ on $\cosh, \sinh$.)

Induced Riemannian metric (first quadratic form) is equal to

$$G_{HyperbolI} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\cosh\theta\cos\varphi, y=a\cosh\theta\sin\varphi, z=a\sinh\theta} =$$

$$(a\sinh\theta\cos\varphi d\theta - a\cosh\theta\sin\varphi d\varphi)^2 + (a\sinh\theta\sin\varphi d\theta + a\cosh\theta\cos\varphi d\varphi)^2 + (a\cosh\theta d\theta)^2 =$$

$$a^2\sinh^2\theta d\theta^2 + a^2\cosh^2\theta d\varphi^2 + a^2\cosh^2\theta d\theta^2 =$$

$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\cosh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} a^2(1+2\sinh^2\theta) & 0 \\ 0 & a^2\cosh^2\theta \end{pmatrix}$$

2) *Two-sheeted hyperboloid*: $z^2 - x^2 - y^2 = a^2$. It is not ruled surface!

For two-sheeted hyperboloid calculations will be very similar.

In the same way as for one-sheeted hyperboloid (see equation (1.1)) it is convenient to choose parameterisation:

$$\mathbf{r}(\theta,\varphi): \qquad \begin{cases} x = a\sinh\theta\cos\varphi \\ y = a\sinh\theta\sin\varphi \\ z = a\cosh\theta \end{cases} \tag{1.2}$$

$$z^2 - x^2 - y^2 = a^2\cosh^2\theta - a^2\sinh^2\theta = a^2$$

(Compare the calculations with calculations for sphere and one-sheeted hyperboloid.

Induced Riemannian metric (first quadratic form) is equal to

$$G_{HyperbolI} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\sinh\theta\cos\varphi, y=a\sinh\theta\sin\varphi, z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 + (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 + a^2\sinh^2\theta d\theta^2 =$$

$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\sinh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} a^2(1+2\sinh^2\theta) & 0 \\ 0 & a^2\sinh^2\theta \end{pmatrix} \tag{1.3}$$

We calculated examples of induced Riemannian structure embedded in Euclidean space almost for all quadratic surfaces.

Quadratic surface is a surface defined by the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + ex + fy + dz + c = 0$$

One can see that any quadratic surface by affine transformation can be transformed to one of these surfaces

- cylinder (elliptic cylinder) $x^2 + y^2 = 1$
- hyperbolic cylinder: $x^2 - y^2 = 1$)
- parabolic cylinder $z = x^2$
- paraboloid $x^2 + y^2 = z$
- hyperbolic paraboloid $x^2 - y^2 = z$
- cone $x^2 + y^2 - z^2 = 0$
- sphere $x^2 + y^2 + z^2 = 1$
- one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$
- two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$

(We exclude degenerate cases such as "point" $x^2 + y^2 + z^2 = 0$, planes, e.t.c.)

1.2 Inversion and metric on circle and sphere in stereographic cooridnates

Formulae (??) and (??) for metric in stereographic coordinates are very important, look very nice, but everyone who tried to calculate them was forced to do difficult calculations. In this paragraph we will explain how these formulae can be derived almost automatically with use of *inversion*

Let O be an arbitrary point in Euclidean space $\mathbf{E}^n$ (Here we consider just the case $n = 2, 3$ [1])

Let S_a be a sphere of radius a with centre at the point 0.

If n^i are coordinates of the point O, then points of the shere are defined by equation $(\sum_{i=1}^{n}(x^i - n^i)^2 = a^2$. We call this sphere base of inversion.

We define inversion of $\mathbf{E}^n$ with respect to the sphere S_a as a map which maps an arbitrary point $P \neq 0$ in $\mathbf{E}^n$ to the point P' such

- point P' belongs to the ray OP
- $$|0P| \cdot ||OP'| = a^2 \tag{1.4}$$

We see that in particular points of the inversion sphere remian fixed under inversion.

It can be proved that transforms lines, k-dimensional, planes, circles,.spheres to lines, planes, circles, spheres, and that the inversion does not change angel between tangent vectors.

Stereographic projection is restriction of inversion.

Hence stereographic projection is conformal map. This is why in stereographic coordinates Riemannian metric has conformal appearance.

1.3 Induced metric on two-sheeted hyperboloid embedded in pseudo-Euclidean space.

Consider the same two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$ embedded $\mathbf{R}^3$ (See equation (1.2). For simplicity we assume now that $a = 1$.) Now we consider the ambient space $\mathbf{R}^3$ not as Euclidean space but as *pseudo-Euclidean space*, i.e. in $\mathbf{R}^3$ instead standard scalar product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = X^1Y^1 + X^2Y^2 + X^3Y^3$$

[1]These considerations can be generalised for arbitrary n

we consider pseudo-scalar product defined by bilinear form

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{pseud} = X^1Y^1 + X^2Y^2 - X^3Y^3$$

The "pseudoscalar" product is bilinear, symmetric. It is defined by non-degenerate matrix. But it is not positive-definite. E.g. The "pseudo-length" of vectors $\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a)$ is equals to zero (such vectors are called null vectors):

$$\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a) \Rightarrow \langle \mathbf{X}, \mathbf{X} \rangle_{pseudo} = 0,$$

The corresponding pseudo-Riemannian metric is:

$$G_{pseudo} = dx^2 + dy^2 - dz^2 \tag{1.5}$$

It turns out that the following remarkable fact occurs:

Proposition *The pseudo-Riemannian metric* (1.5) *in the ambient* 3*-dimensional pseudo-Euclidean space induces Riemannian metric on two-sheeted hyperboloid* $x^2 + y^2 - z^2 = 1$.

Remark This is not the fact for one-sheeted hyperboloid (see problem 7 in Homework 2)

Show it. (See also problems 5 and 6 in Homework 2.) Repeat the calculations above for two-sheeted hyperboloid changing in the ambient space Riemannian metric $G = dx^2 + dy^2 + dz^2$ on pseudo-Riemannian $dx^2 + dy^2 - dz^2$:

Using (1.2) and (1.5) we come now to

$$G = \left(dx^2 + dy^2 - dz^2\right)\big|_{x=a\sinh\theta\cos\varphi, y=a\sinh\theta\sin\varphi, z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 - (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 - a^2\sinh^2\theta d\theta^2$$

$$, \qquad G_L = a^2 d\theta^2 + a^2\sinh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix} \tag{1.6}$$

The two-sheeted hyperboloid equipped with this metric is called hyperbolic or Lobachevsky plane.

Now express Riemannian metric in stereographic coordinates. (We did it in detail in homework 2)

Calculations are very similar to the case of stereographic coordinates of 2-sphere $x^2 + y^2 + z^2 = 1$. (See homework 1). Centre of projection $(0, 0, -1)$: For stereographic coordinates u, v we have $\frac{u}{x} = \frac{y}{v} = \frac{1}{1+z}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}, \qquad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases} \tag{4}$$

The image of upper-sheet is an open disc $u^2+v^2=1$ since $u^2+v^2=\frac{x^2+y^2}{(1+z)^2}=\frac{z^2-1}{(1+z)^2}=\frac{z-1}{z+1}$. Since for upper sheet $z>1$ then $0\le\frac{z-1}{z+1}<1$.

$$G=(dx^2+dy^2-dz^2)\big|_{x=x(u,v),y=y(u,v),z=z(u,v)}=\left(d\left(\frac{2u}{1-u^2-v^2}\right)\right)^2+$$

$$\left(d\left(\frac{2v}{1-u^2-v^2}\right)\right)^2-\left(d\left(\frac{u^2+v^2+1}{1-u^2-v^2}\right)\right)^2=\frac{4(du)^2+4(dv)^2}{(1-u^2-v^2)^2}\,.$$

These coordinates are very illuminating. One can show that we come to so called hyperbolic plane (see in detail Homework 2)

2 Isometries and infinitesimal isometries (Killing vector fields)

Let $\mathbf{X}$ be an arbitrary vector field on Riemannian manifold M. It induces infinitesimal diffeomorphism

$$F\colon x^{i'}=x^i+\varepsilon X^i(x),\quad \text{where } \varepsilon^2=0\,.$$

(the condition $\varepsilon^2=0$ reflects the fact that we ignore terms of order ≥ 2 over ε.) Find a condition which guarantees that infinitesimal diffeomorphism is an isometry. If $x^{i'}=x^i+\varepsilon X^i(x)$, then one can see that the inverse infinitesimal diffeomorphism is defined by the equation $x^i=x^{i'}-\varepsilon X^i(x')$ and equation (**??**) implies that

$$g_{ik}(x)=g_{pq}(x'(x))\frac{\partial x^p(x')}{\partial x^i}\frac{\partial x^q(x')}{\partial x^k}=g_{pq}(x^i{+}\varepsilon X^i)\left(\delta^p_i+\varepsilon\frac{pX^p(x)}{\partial x^i}\right)\left(\delta^q_k+\varepsilon\frac{pX^q(x)}{\partial x^k}\right)=$$

$$g_{ik}(x)+\varepsilon\left[X^p(x)\frac{\partial g_{ik}(x)}{\partial x^p}+g_{iq}(x)\frac{\partial X^q(x)}{\partial x^k}+g_{pk}(x)\frac{\partial X^p(x)}{\partial x^i}\,,\right]$$

Here we consider only terms of first and zero order over ε since $\varepsilon^2=0$ (this is related with the fact that transformation is *infinitesimal*). The last relation implies that

$$X^p(x)\frac{\partial g_{ik}(x)}{\partial x^p}+g_{iq}(x)\frac{\partial X^q(x)}{\partial x^k}+g_{pk}(x)\frac{\partial X^p(x)}{\partial x^i}=0\,. \tag{2.1}$$

Left hand side of this relation we denote $\mathcal{L}_{\mathbf{X}}G$— Lie derivative of Riemannian metric along vector field $\mathbf{X}$. Vector field $\mathbf{X}$ induces isometry if Lie derivative of metric along this vector field vanishes. We come to

Proposition Vector field $\mathbf{X}$ on Riemannian manifold (M,G) induces infinitesimal isometry if $\mathcal{L}_{\mathbf{X}}G=0$:

$$\mathcal{L}_{\mathbf{X}}G = X^p(x)\frac{\partial g_{ik}(x)}{\partial x^p} + g_{iq}(x)\frac{\partial X^q(x)}{\partial x^k} + g_{pk}(x)\frac{\partial X^p(x)}{\partial x^i} = 0\,. \qquad (2.2)$$

Definition) We call vector field $\mathbf{X}$ *Killing vector field)* if it preserves the metric, i.e. if equation (2.2) is obeyed.

Example Consider plane (x,y) with Riemannian metric $G=\sigma(x,y)(dx^2+dy^2)$. Find differential equation for infinitesimal isometries of this metric, i.e. write down equations (2.2) for this metric.

We have $||g_{ik}(x,y)|| = \begin{pmatrix}\sigma(x,y) & 0\\ 0 & \sigma(x,y)\end{pmatrix}$.

Let $\mathbf{X}=A(x,y)\partial_x+B(x,y)\partial_y$. Write down equations (2.2) for components g_11, g_12, g_{21} and g_{22}: We will have the following three equations

$$\begin{cases} A(x,y)\frac{\partial\sigma}{\partial x}+B(x,y)\frac{\partial\sigma}{\partial y}+2\frac{\partial A(x,y)}{\partial x}\sigma=0 \quad \text{for component } g_{11}\\ A(x,y)\frac{\partial\sigma}{\partial x}+B(x,y)\frac{\partial\sigma}{\partial y}+2\frac{\partial B(x,y)}{\partial y}\sigma=0 \quad \text{for component } g_{22}\\ \frac{\partial B(x,y)}{\partial x}+\frac{\partial A(x,y}{\partial y}=0 \quad \text{for components } g_{12} \text{ and } g_{21}\end{cases} \qquad (2.3)$$

Practically for sphere, Lobachevsky plane, e.t.c. it is much easier to find the Killing fields not solving these equations, but considering the usual isometries (see examples in solutions of Coursework and in the Appendix about Killing vector fields for Lobachevsky plane.))

Another simple and interesting exercise: How look Killing vectors for Euclidean space $\mathbf{E}^n$. In this case we come from (2.2) to equation

$$\mathcal{L}_{\mathbf{K}}G = \delta_{iq}(x)\frac{\partial K^q(x)}{\partial x^k} + \delta_{pk}(x)\frac{\partial K^p(x)}{\partial x^i} = 0\,,$$

i.e.

$$\frac{\partial K^i(x)}{\partial x^k} + \frac{\partial K^k(x)}{\partial x^i} = 0\,. \qquad (2.4)$$

Solve this equation. Differentiating by x we come to

$$\frac{\partial^2 K^i(x)}{\partial x^m\partial x^k} + \frac{\partial K^k(x)}{\partial x^m\partial x^i} = 0$$

Consider tensor field

$$T^i_{mk} = \frac{\partial^2 K^i}{\partial x^m\partial x^k} \qquad (2.5)$$

It follows from equation (2.4) that

$$T^i_{mk} = T^i_{km} = -T^m_{ik}\,. \qquad (2.6)$$

It is easy to see that this implies that $T^i_{mk} \equiv 0$!!!:

$$T^i_{mk} = -T^m_{ik} = -T^m_{ki} = T^k_{mi} = T^k_{im} = -T^i_{km} = -T^i_{mk} \Rightarrow T^i_{mk} = -T^i_{mk}\,,$$

i.e. $T^i_{mk} = \frac{\partial^2 K^i(x)}{\partial x^m \partial x^k} = 0$. This implies that

$$K^i(x) = C^i + B^i_k x^k$$

We come to

Theorem All infinitesimal isometries of $\mathbf{E}^n$ are translations and infinitesimal rotations.

What happens in general case?

3 Invariance of volume element under changing of coordinates

Check straighforwardly that volume element is invariant under coordinate transformations, i.e. if $y^1, \ldots, y^n$ are new coordinates: $x^1 = x^1(y^1, \ldots, y^n), x^2 = x^2(y^1, \ldots, y^n)$...,

$$x^i = x^i(y^p), i = 1, \ldots, n \quad , p = 1, \ldots, n$$

and $\tilde{g}_{pq}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q}\,. \tag{3.1}$$

Then

$$\sqrt{\det g_{ik}(x)}\, dx^1 dx^2 \ldots dx^n = \sqrt{\det \tilde{g}_{pq}(y)}\, dy^1 dy^2 \ldots dy^n \tag{3.2}$$

This follows from (3.1). Namely

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \ldots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)}\, dy^1 dy^2 \ldots dy^n$$

Using the fact that $\det(ABC) = \det A \cdot \det B \cdot \det C$ and $\det\left(\frac{\partial x^i}{\partial y^p}\right) = \det\left(\frac{\partial x^k}{\partial y^q}\right)$[2] we see that from the formula above follows:

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \ldots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)} dy^1 dy^2 \ldots dy^n =$$

[2] determinant of matrix does not change if we change the matrix on the adjoint, i.e. change columns on rows.

$$\sqrt{\left(\det\left(\frac{\partial x^i}{\partial y^p}\right)\right)^2}\sqrt{\det g_{ik}(x(y))}dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\det g_{ik}(x(y))}\det\left(\frac{\partial x^i}{\partial y^p}\right) dy^1 dy^2 \dots dy^n = \tag{3.3}$$

Now note that

$$\det\left(\frac{\partial x^i}{\partial y^p}\right) dy^1 dy^2 \dots dy^n = dx^1 \dots dx^n$$

according to the formula for changing coordinates in n-dimensional integral [3]. Hence

$$\sqrt{\det g_{ik}(x(y))}\det\left(\frac{\partial x^i}{\partial y^p}\right) dy^1 dy^2 \dots dy^n = \sqrt{\det g_{ik}(x(y))}dx^1 dx^2 \dots dx^n \tag{3.4}$$

Thus we come to (3.2).

4 Connection

4.1 Global aspects of existence of connection

We defined connection as an operation on vector fields obeying the special axioms (see the subsubsection 2.1.1). Then we showed that in a given coordinates connection is defined by Christoffel symbols. On the other hand we know that in general coordinates on manifold are not defined globally. (We had not this trouble in Euclidean space where there are globally defined Cartesian coordinates.)

- How to define connection globally using local coordinates?
- Does there exist at least one globally defined connection?
- Does there exist globally defined flat connection?

These questions are not naive questions. Answer on first and second questions is "Yes". It sounds bizzare but answer on the first question is not "Yes" [4]

[3] Determinant of the matrix $\left(\frac{\partial x^i}{\partial y^p}\right)$ of changing of coordinates is called sometimes Jacobian. Here we consider the case if Jacobian is positive. If Jacobian is negative then formulae above remain valid just the symbol of modulus appears.

[4] Topology of the manifold can be an obstruction to existence of global flat connection. E.g. it does not exist on sphere S^n if $n > 1$.

Global definition of connection

The formula (**??**) defines the transformation for Christoffer symbols if we go from one coordinates to another.

Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold M.

If connection ∇ is defined on the manifold M then it defines in any chart (local coordinates) (x^i_α) Christoffer symbols which we denote by ${}_{(\alpha)}\Gamma^i_{km}$. If (x^i_α), $(x^{i'}_{(\beta)})$ are different local coordinates in a vicinity of a given point then according to (**??**)

$$
{}_{(\beta)}\Gamma^{i'}_{k'm'} = \frac{\partial x^k_{(\alpha)}}{\partial x^{k'}_{(\beta)}} \frac{\partial x^m_{(\alpha)}}{\partial x^{m'}_{(\beta)}} \frac{\partial x^{i'}_{(\beta)}}{\partial x^i_{(\alpha)}}{}_{(\beta)} \Gamma^{(\alpha)i}_{mk} + \frac{\partial^2 x^k_{(\alpha)}}{\partial x^{m'}_{(\beta)} \partial x^{k'}_{(\beta)}} \frac{\partial x^{i'}_{(\beta)}}{\partial x^k_{(\alpha)}} \tag{4.1}
$$

Definition Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold M

We say that the collection of Christoffel symbols $\{\Gamma^{(\alpha)i}_{km}\}$ defines globally a connection on the manifold M in this atlas if for every two local coordinates $(x^i_{(\alpha)})$, $(x^i_{(\beta)})$ from this atlas the transformation rules (4.1) are obeyed.

Using partition of unity one can prove the existence of global connection constructing it in explicit way. Let $\{(x^i_\alpha), U_\alpha\}$ $(\alpha = 1, 2, \ldots, N)$ be a finite atlas on the manifold M and let $\{\rho_\alpha\}$ be a partition of unity adjusted to this atlas. Denote by ${}^{(\alpha)}\Gamma^i_{km}$ local connection defined in domain U_α such that its components in these coordinates are equal to zero. Denote by ${}^{(\alpha)}_{(\beta)}\Gamma^i_{km}$ Christoffel symbols of this local connection in coordinates $(x^i_{(\beta)})$ $({}^{(\alpha)}_{(\beta)}\Gamma^i_{km} = 0)$. Now one can define globally the connection by the formula:

$$
{}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x})\, {}^{(\alpha)}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x}) \frac{\partial x^i_{(\beta)}}{\partial x^{i'}_{(\alpha)}} \frac{\partial^2 x^{i'}_{(\alpha)}(\mathbf{x})}{\partial x^k_{(\beta)} \partial x^m_{(\beta)}}\,. \tag{4.2}
$$

This connection in general is not flat connection[5]

4.2 Killing vectors, antisymmetric operator and antisymmetric bilinear form

We return to Killing vectors.

First consider the following construction.

Let $\mathbf{K}$ be an arbitrary vector field, then consider bilinear form

$$
S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) = G\left(\nabla_{\mathbf{X}}\mathbf{K}\,, \mathbf{Y}\right) = \langle \nabla_{\mathbf{X}}\mathbf{K}\,, \mathbf{Y}\rangle = \langle \mathbf{Y}\,, \nabla_{\mathbf{X}}\mathbf{K}\rangle\,, \tag{4.3}
$$

[5]See for detail the text: "Global affine connection on manifold" " in my homepage: "www.maths.mancheser.ac.uk/khudian" in subdirectory Etudes/Geometry

where ∇ is an arbitrary connection, G Riemannian metric, defining scalar product $\langle\ ,\ \rangle$, and $\mathbf{K}$,$\mathbf{X}, \mathbf{Y}$ arbitrary vector fields. One can see that for arbitrary functions f, g

$$S(f\mathbf{X}, g\mathbf{Y}) = fgS(\mathbf{X}, \mathbf{Y})$$

This immediately follows from definition of connection (see condition (**??**) in **??**). In local coordinates if $\mathbf{X} = X^m\partial_m$, $\mathbf{Y} = Y^k\partial_k$ then

$$S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) = S(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) = S(\nabla_{X^m\partial_m}K^i\partial_i, Y^n\partial_n) = X^mY^nS_{mn}\,,$$

where

$$S_{mn} = \langle(\partial_mK^i+\Gamma^i_{mr}K^r)K^r\partial_i, \partial_n\rangle = \langle(\partial_mK^i+\Gamma^i_{mr}K^r)\partial_i, \partial_n\rangle = \left(\partial_mK^i + \Gamma^i_{mr}K^r\right)g_{in}\,. \tag{4.4}$$

We see that this construction defines covariant tensor field.

Theorem Let ∇ be Levi-Civita connection on Riemannian manifold (M, G). Then vector field $\mathbf{K}$ is a Killing vector field on M if and only if tensor field $S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y})$ is antisymmetric tensor field: $S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) = -S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y})$.

Proof

First recall properties of Killing vector field.

Let M be Riemannian manifold with Riemannian metric G. Recall that a vector field $\mathbf{K}$ is Killing vector field, i.e. it defines infinitesimal isometry, if

$$\mathcal{L}_{\mathbf{K}}G = 0\,,$$

Notice that for an arbitrary vector field $\mathbf{Z}$ and arbitrary vetor fields $\mathbf{X}, \mathbf{Y}$ we have

$$\mathcal{L}_{\mathbf{Z}}\left(G\left(\mathbf{X}, \mathbf{Y}\right)\right) = \partial_{\mathbf{Z}}G(\mathbf{X}, \mathbf{Y}) =$$

$$= (\mathcal{L}_{\mathbf{Z}}G)(\mathbf{X}, \mathbf{Y})+G(\mathcal{L}_{\mathbf{Z}}\mathbf{X}, \mathbf{Y})+G(\mathbf{X}, \mathcal{L}_{\mathbf{Z}}\mathbf{Y}) = \mathcal{L}_{\mathbf{Z}}G(\mathbf{X}, \mathbf{Y})+G([\mathbf{Z}, \mathbf{X}], \mathbf{Y})+G(\mathbf{X}, [\mathbf{Z}, \mathbf{Y}])\,. \tag{4.5}$$

. In the case if $\mathbf{Z} = \mathbf{K}$ is Killing vector field then condition $\mathcal{L}_{\mathbf{K}}G = 0$ and

$$\partial_{\mathbf{K}}G(\mathbf{X}, \mathbf{Y}) = G(\mathcal{L}_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \mathcal{L}_{\mathbf{K}}\mathbf{Y}) = G([\mathbf{K}, \mathbf{X}], \mathbf{Y}) + G(\mathbf{X}, [\mathbf{K}, \mathbf{Y}])\,. \tag{4.6}$$

Now let ∇ be Levi-Civita connection of Riemannian metric, i.e.

$$\partial_{\mathbf{Z}}G(\mathbf{X}, \mathbf{Y}) = G(\nabla_{\mathbf{Z}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{Z}}\mathbf{Y})$$

for arbitrary vector fields $\mathbf{Z}, \mathbf{X}$,
Y (see levicivitaconnection1). In particular for $\mathbf{Z} = \mathbf{K}$ we have

$$\partial_{\mathbf{K}}G(\mathbf{X}, \mathbf{Y}) = G(\nabla_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{K}}\mathbf{Y})\,.$$

Now transform the relation (4.6) and compare it with this relation. Performing this transformation we will use the symmetricity of Levi-Civita connection, i.e.

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}] . \tag{4.7}$$

(see symmetricconnectioninvariant and Homework 4.) We have:

$$\partial_{\mathbf{K}} G(\mathbf{X}, \mathbf{Y}) = G(\mathcal{L}_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \mathcal{L}_{\mathbf{K}}\mathbf{Y}) = G([\mathbf{K}, \mathbf{X}], \mathbf{Y}) + G(\mathbf{X}, [\mathbf{K}, \mathbf{Y}]) =$$

$$G(\nabla_{\mathbf{K}}\mathbf{X} - \nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{K}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{K}) =$$

$$\underbrace{G(\nabla_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{K}}\mathbf{Y})}_{\partial_{\mathbf{K}}(\mathbf{X},\mathbf{Y})} - G(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) - G(\mathbf{X}, \nabla_{\mathbf{Y}}\mathbf{K})$$

This implies that

$$G(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{Y}}\mathbf{K}) = S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) + S_{\mathbf{K}}(\mathbf{Y}, \mathbf{X}) = 0 .$$

, i.e. $S_{\mathbf{K}}$ is antisymmetric. One can easy to see that converse implication is also true.

Remark Note that the antisymmetric tensor field $S_{\mathbf{K}}$ defines antisymmetric linear operator

$$A\colon \quad \mathbf{X} \mapsto \nabla_{\mathbf{X}}\mathbf{K} .$$

5 Geodesics and Lagrangians

5.1 Variational principe and Euler-Lagrange equations

Here very briefly we will explain how Euler-Lagrange equations follow from variational principe.

Let M be a manifold (not necessarily Riemannian) and $L = L(x^i, \dot{x}^i)$ be a Lagrangian on it.

Denote my $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$ the space of curves (paths) such that they start at the point $\mathbf{x}_1$ at the "time" $t = t_1$ and end at the point $\mathbf{x}_2$ at the "time" $t = t_2$:

$$\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} = \{C\colon\ \mathbf{x}(t), t_1 \le t \le t_2,\ \mathbf{x}(t_1) = \mathbf{x}_1, \mathbf{x}(t_2) = \mathbf{x}_2\} . \tag{5.1}$$

Consider the following functional S on the space $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$:

$$S[\mathbf{x}(t)] = \int_{t_1}^{t_2} L\left(x^i(t), \dot{x}^i(t)\right) dt . \tag{5.2}$$

for every curve $\mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$.

This functional is called *action* functional.

Theorem *Let functional S attaints the minimal value on the path $\mathbf{x}_0(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$, i.e.*

$$\forall \mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} \quad S[\mathbf{x}_0(t)] \leq S[\mathbf{x}(t)]\,. \tag{5.3}$$

Then the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \text{ if } \mathbf{x}(t) = \mathbf{x}_0(t)\,. \tag{5.4}$$

Remark The path $\mathbf{x}(t)$ sometimes is called *extremal* of the action functional (5.2).

We will use this Theorem to show that the geodesics are in some sense shortest curves [6].

5.2 Geodesics and shortest distance.

Many of you know that geodesics are in some sense shortest curves. We will give an exact meaning to this statement and prove it using variational principe:

Let M be a Riemannian manifold.

Theorem *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points on M. The shortest curve which joins these points is an arc of geodesic.*

Let C be a geodesic on M and $\mathbf{x}_1 \in C$. Then for an arbitrary point $\mathbf{x}_2 \in C$ which is close to the point $\mathbf{x}_1$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points[7] .

This Theorem makes a bridge between two different approach to geodesic: the shortest disntance and parallel transport of velocity vector.

Sketch a proof:

[6]The statement of this Theorem is enough for our purposes. In fact in classical mechanics another more useful statement is used: the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian L if and only if it is the stationary "point" of the action functional (5.2), i.e.

$$S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] - S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] = 0(\delta\mathbf{x}(t)) \tag{5.5}$$

for an arbitrary infinitesimal variation of the path $\mathbf{x}_0(t)$: $\delta\mathbf{x}_(t_1) = \delta\mathbf{x}_(t_2) = 0$.

[7]More precisely: for every point $\mathbf{x}_1 \in C$ there exists a ball $B_\delta(\mathbf{x}_1)$ such that for an arbitrary point $\mathbf{x}_2 \in C \cap B_\delta(\mathbf{x}_1)$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points.

Consider the following two Lagrangians: Lagrangian of a "free " particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ and the length Lagrangian

$$L_{\text{length}}(x,\dot{x}) = \sqrt{g_{ik}(x)\dot{x}^i\dot{\mathbf{x}}^k} = \sqrt{2L_{\text{free}}}\,.$$

If $C\colon\ x^i(t), t_1 \le t \le t_2$ is a curve on M then

$$\text{Length of the curve } C =$$

$$\int_{t_1}^{t_2} L_{\text{length}}(x^i(t),\dot{x}^i(t))dt = \int_{t_1}^{t_2}\sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt\,. \tag{5.6}$$

The proof of the Theorem follows from the following observation:

Observation Euler-Lagrange equations for the length functional (5.6) are equivalent to the Euler-Lagrange equations for action functional (5.2). This means that extremals of the length functional and action functionals coincide.

Indeed it follows from this observation and the variational principe that the shortest curves obey the Euler-Lagrange equations for the action functional. We showed before that Euler-Lagrange equations for action functional (5.2) define geodesics. Hence the shortest curves are geodesics.

One can check the observation by direct calculation: Calculate Euler-Lagrange equations for the Lagrangian $L_{\text{length}} = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2L_{\text{free}}}$:

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{d}{dt}\left(\frac{1}{\sqrt{g_{ik}\dot{x}^i\dot{x}^k}}g_{ik}\dot{x}^k\right) - \frac{1}{2\sqrt{g_{ik}\dot{x}^i\dot{x}^k}}\frac{\partial g_{km}\dot{x}^k\dot{x}^m}{\partial x^i}$$

$$= \frac{d}{dt}\left(\frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\partial \dot{x}^i}\right) - \frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\partial x^i} = 0\,. \tag{5.7}$$

To facilitate calculations note that the length functional (5.6) is reparameterisation invariant. Choose the natural parameter $s(t)$ or a parameter proportional to the natural parameter on the curve $x^i(t)$. We come to $L_{\text{length}} = const$ and it follows from (5.7) that

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{1}{L_{\text{length}}}\left(\frac{d}{dt}\left(\frac{\partial L_{\text{free}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{free}}}{\partial x^i}\right) = 0\,.$$

We prove that Euler-Lagrange equations for length and action Lagrangians coincide. ■

In the Euclidean space straight lines are the shortest distances between two points. On the other hand their velocity vectors are constant. We realise now that in general Riemannian manifold the role of geodesic is twofold also: they are locally shortest and have covariantly constant velocity vectors.

5.2.1 Again geodesics for sphere and Lobachevsky plane

The fact that geodesics are shortest gives us another tool to calculate geodesics.

Consider again examples of sphere and Lobachevsky plane and find geodesics using the fact that they are shortest. The fact that geodesics are locally the shortest curves

Consider again sphere in $\mathbf{E}^3$ with the radius R: Coordinates θ, φ, induced Riemannian metrics (first quadratic form):

$$G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{5.8}$$

Consider two arbitrary points A and B on the sphere. Let (θ_0, φ_0) be coordinates of the point A and (θ_1, φ_1) be coordinates of the point B

Let C_{AB} be a curve which connects these points: $C_{AB}\colon \theta(t), \varphi(t)$ such that $\theta(t_0) = \theta_0, \theta(t_1) = \theta_1$, $\varphi(t_0) = \theta_0, \theta(t_1) = \theta_1$ then:

$$L_{C_{AB}} = \int R\sqrt{\theta_t^2 + \sin^2\theta(t)\varphi_t^2}dt \tag{5.9}$$

Suppose that points A and B have the same latitude, i.e. if (θ_0, φ_0) are coordinates of the point A and (θ_1, φ_1) are coordinates of the point B then $\varphi_0 = \varphi_1$ (if it is not the fact then we can come to this condition rotating the sphere)

Now it is easy to see that an arc of meridian, the curve $\varphi = \varphi_0$ is geodesics: Indeed consider an arbitrary curve $\theta(t), \varphi(t)$ which connects the points A, B: $\theta(t_0) = \theta(t_1) = \theta_0$, $\varphi(t_0) = \varphi(t_1) = \varphi_0$. Compare its length with the length of the meridian which connects the points A, B:

$$\int_{t_0}^{t_1} R\sqrt{\theta_t^2 + \sin^2\theta\varphi_t^2}dt \geq R\int_{t_0}^{t_1}\sqrt{\theta_t^2}dt = R\int_{t_0}^{t_1}\theta_t dt = R(\theta_1 - \theta_0) \tag{5.10}$$

Thus we see that the great circle joining points A, B is the shortest. *The great circles on sphere are geodesics.* It corresponds to geometrical intuition: The geodesics on the sphere are the circles of intersection of the sphere with the plane which crosses the centre.

Geodesics on Lobachevsky plane

Riemannian metric on Lobachevsky plane:

$$G = \frac{dx^2 + dy^2}{y^2} \tag{5.11}$$

The length of the curve $\gamma\colon x = x(t, y = y(t))$ is equal to

$$L = \int\sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}}dt$$

In particularly the length of the vertical interval $[1,\varepsilon]$ tends to infinity if $\varepsilon \to 0$:

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt = \int_\varepsilon^1 \sqrt{\frac{1}{t^2}} dt = \log \frac{1}{\varepsilon}$$

One can see that the distance from every point to the line $y = 0$ is equal to infinity. This motivates the fact that the line $y = 0$ is called *absolute*.

Consider two points $A = (x_0, y_0)$, $B = (x_1, y_1)$ on Lobachevsky plane.

It is easy to see that vertical lines are geodesics of Lobachevsky plane.

Namely let points A, B are on the ray $x = x_0$. Let C_{AB} be an arc of the ray $x = x_0$ which joins these points: $C_{AB}\colon x = x_0, y = y_0 + t$ Then it is easy to see that the length of the curve C_{AB} is less or equal than the length of the arbitrary curve $x = x(t), y = y(t)$ which joins these points: $x(t)\big|_{t=0} = x_0, y(t)\big|_{t=0} = y_0$, $x(t)\big|_{t=t_1} = x_0, y(t)\big|_{t=t_1} = y_1$:

$$\int_0^t \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt \geq \int_0^t \sqrt{\frac{y_t^2}{y^2(t)}} dt = \int_{y_0}^{y_1} \frac{dt}{t} dt = \log \frac{y_1}{y_0} = \text{length of } C_{AB}$$

Hence C_{AB} is shortest. We prove that vertical rays are geodesics.

Consider now the case if $x_0 \neq x_1$. Find geodesics which connects two points A, B which are not on the same vertical ray. Consider semicircle which passes these two points such that its centre is on the absolute. We prove that it is a geodesic.

Proof Let coordinates of the centre of the circle are $(a, 0)$. Then consider polar coordinates (r, φ):

$$x = a + r\cos\varphi, y = r\sin\varphi \tag{5.12}$$

In these polar coordinates r-coordinate of the semicircle is constant.

Find Lobachevsky metric in these coordinates: $dx = -r\sin\varphi d\varphi + \cos\varphi dr$, $dy = r\cos\varphi d\varphi + \sin\varphi dr$, $dx^2 + dy^2 = dr^2 + r^2 d\varphi^2$. Hence:

$$G = \frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\varphi^2}{r^2 \sin^2\varphi} == \frac{d\varphi^2}{\sin^2\varphi} + \frac{dr^2}{r^2\sin^2\varphi} \tag{5.13}$$

We see that the length of the arbitrary curve which connects points A, B is greater or equal to the length of the arc of the circle:

$$L_{AB} = \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi} + \frac{r_t^2}{r^2\sin^2\varphi}} dt \geq \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi}} dt = \tag{5.14}$$

$$\int_{t_0}^{t_1} \frac{\varphi_t}{\sin\varphi} dt = \int_{\varphi_0}^{\varphi_1} \frac{d\varphi}{\sin\varphi} = \log\frac{\tan\varphi_1}{\tan\varphi_1}$$

The proof is finished.

5.3 Integrals of motions and geodesics.

5.3.1 Magnitudes preserving along geodesics—Integrals of motion

It is very useful to find magnitudes which are preserved along geodesics, functions $F = F(x, \dot{x})$ such that for geodesic $C\colon x^i = x^i(t)$ the magnitude

$$I(t) = F(x, \dot{x})\big|\_x^i = x^i(t) \quad \text{is preserved along geodesic } x^i(t), \quad \frac{dI(t)}{dt} = 0\,. \tag{5.15}$$

Geodesics are solutions of equations of motions for the Lagrangian of a free particle $L = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$. One can consider such functions $F = F(x, \dot{x})$ for an arbitrary Lagrangian L. In this case $x^i(t)$ is a solution of the Lagrangian L.

These magnitudes which are preserved along solutions of equations of motion (in particular along geodesics in the case if L is the Lagrangian of a free particle) are called *integrals of motion* (See in detail about integrals of motion in Appendix to this lectures).

There is the following very useful criterion to find magnitudes, which are preserved on equations of motions, i.e. integrals fo motion.

Proposition *Let Lagrangian $L(x^i, \dot{x}^i)$ in coordinates $\{x^i\}$ does not depend, say on the coordinate x^1. $L = L(x^2, \dots, x^n, \dot{x}^1, \dot{x}^2, \dots, \dot{x}^n)$. Then the function*

$$F_1(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^1}$$

is integral of motion. (In the case if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i. the function $F_i(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i}$ will be integral of motion.)

Proof is simple. Check the condition (5.15): Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (i = 1, 2, \dots, n)$$

In particular for first coordinate x^1, $\frac{\partial L}{\partial x^1} = 0$ and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) - \frac{\partial L}{\partial x^1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = 0\,,$$

i.e. the magnitude $I(t) = F(x, \dot{x})$ is preserved if $F = \frac{\partial L}{\partial \dot{x}^1}$. We see that exactly first equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = \frac{d}{dt} F_1(q, \dot{q}) = 0 \quad \text{since } \frac{\partial L}{\partial x^1} = 0,.$$

(if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i then the function $F_i(x, \dot{x}) = \frac{\partial L}{\dot{x}^1}$ is integral of motion since $i - th$ equation is exactly the condition $\dot{F}_i = 0$.)

The integral of motion $F_i = \frac{\partial L}{\partial \dot{x}^i}$ is called sometimes *generalised momentum.*

Consider examples of calculation of preserved mangnitudes along geodesics.

Example (sphere)

Sphere of the radius R in $\mathbf{E}^3$. Riemannian metric: $G = Rd\theta^2 + R^2 \sin^2\theta d\varphi^2$ and $L_{\text{free}} = \frac{1}{2}\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2\right)$. Lagrangian does not depend explicitly on coordinate φ. The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = R^2 \sin^2\theta \dot{\varphi}\,.$$

It is preserved along geodesics, i.e. along great circles.

Example (cone)

Consider cone $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$. Riemannian metric:

$$G = d(kh\cos\varphi)^2 + d(kh\sin\varphi)^2 + (dh)^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2\,.$$

and free Lagrangian

$$L_{\text{free}} = \frac{(k^2+1)\dot{h}^2 + k^2h^2\dot{\varphi}^2}{2}\,.$$

Lagrangian does not depend explicitly on coordinate h. The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = k^2h^2\dot{\varphi}.$$

It is preserved along geodesics.

Remark One has to note that for the Lagrangian of a free particle $F = L = g_{ik}\dot{x}^i\dot{x}^k$, kinetik energy, is integral of motion preserved along geodesic:it is nothing that square of the length of velocity vector which is preserved along the geodesic.

5.3.2 Using integral of motions to calculate geodesics

Integrals of motions may be very useful to calculate geodesics. The equations for geodesics are second order differential equations. If we know integrals of motions they help us to solve these equations. Consider just an example.

For Lobachevsky plane the free Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$. We already calculated geodesics in the subsection 3.3.4. Geodesics are solutions of second order Euler-Lagrange equations for the Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ (see the subsection 3.3.4)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \end{cases}$$

It is not so easy to solve these differential equations.

For Lobachevsky plane we know two integrals of motions:

$$E = L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}, \quad \text{and } F = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}\,. \tag{5.16}$$

These both integrals are preserved in time: if $x(t), y(t)$ is geodesics then

$$\begin{cases} F = \frac{\dot{x}(t)}{y(t)^2} \\ E = \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{2y(t)^2} = C_2 \end{cases} \Rightarrow \begin{cases} \dot{x} = C_1 y^2 \\ \dot{y} = \pm\sqrt{2C_2 y^2 - C_1^2 y^4} \end{cases} \tag{5.17}$$

These are first order differential equations. It is much easier to solve these equations in general case than initial second order differential equations.

We see how useful in Riemannian geometry to use the Lagrangian approach.

To solve and study solutions of Lagrangian equations (in particular geodesics which are solutions of Euler-Lagrange equations for Lagrangian of free particle) it is very useful to use integrals of motion

5.3.3 Integral of motion for arbitrary Lagrangian $L(x, \dot{x})$

Let $L = L(x, \dot{x})$ be a Lagrangian, the function of point and velocity vectors on manifold M (the function on tangent bundle TM).

Definition We say that the function $F = F(q, \dot{q})$ on TM is *integral of motion* for Lagrangian $L = L(x, \dot{x})$ if for any curve $q = q(t)$ which is the solution of Euler-Lagrange equations of motions the magnitude $I(t) = F(x(t), \dot{x}(t))$ is preserved along this curve:

$$F(x(t), \dot{x}(t)) = const \text{ if } x(t) \text{ is a solution of Euler-Lagrange equations(\textbf{??})}. \tag{5.18}$$

In other words

$$\frac{d}{dt}\left(F(x(t), \dot{x}(t))\right) = 0 \text{ if } x^i(t)\colon\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0\,. \tag{5.19}$$

5.3.4 Basic examples of Integrals of motion: Generalised momentum and Energy

Let $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^1. $L = L(x^2, \ldots, x^n, \dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n)$. Then the function

$$F_1(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^1}$$

is integral of motion. (In the case if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i. the function $F_i(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i}$ will be integral of motion.)

Proof is simple. Check the condition (5.19): Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (i = 1, 2, \dots, n)$$

We see that exactly first equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = \frac{d}{dt}F_1(q, \dot{q}) = 0 \quad \text{since } \tfrac{\partial L}{\partial x^1} = 0, .$$

(if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i then the function $F_i(x, \dot{x}) = \frac{\partial L}{\dot{x}^1}$ is integral of motion since $i - th$ equation is exactly the condition $\dot{F}_i = 0$.)

The integral of motion $F_i = \frac{\partial L}{\partial \dot{x}^i}$ is called sometimes *generalised momentum.*

Another very important example of integral of motion is: energy.

$$E(x, \dot{x}) = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\,. \tag{5.20}$$

One can check by direct calculation that it is indeed integral of motion. Using Euler Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}$ we have:

$$\frac{d}{dt}E(x(t), \dot{x}(t)) = \frac{d}{dt}\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\right) = \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i - \frac{dL}{dt} =$$

$$\frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} - \frac{dL(x, \dot{x})}{dt} = \frac{dL(x, \dot{x})}{dt} - \frac{dL(x, \dot{x})}{dt} = 0\,.$$

5.3.5 Integrals of motion for geodesics

Apply the integral of motions for studying geodesics.

The Lagrangian of "free" particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$. For Lagrangian of free particle solution of Euler-Lagrange equations of motions are geodesics.

If $F = F(x, \dot{x})$ is the integral of motion of the free Lagrangian $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ then the condition (5.18) means that the magnitude $I(t) = F(x^i(t), \dot{x}^i(t))$ is preserved along the geodesics:

$$I(t) = F(x^i(t), \dot{x}^i(t)) = const, \text{i.e. } \frac{d}{dt}I(t) = 0 \;\text{ if } x^i(t) \text{ is geodesic.} \tag{5.21}$$

Consider examples of integrals of motion for free Lagrangian, i.e. magnitudes which preserve along the geodesics:

Example 1 Note that for an arbitrary "free" Lagrangian Energy integral (5.20) is an integral of motion:

$$E = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L = \dot{x}^i \frac{\partial \left(\frac{g_{pq}(x)\dot{x}^p\dot{x}^q}{2} \right)}{\partial \dot{x}^i} - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} =$$

$$\dot{x}^i g_{iq}(x)\dot{x}^q - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} \,. \tag{5.22}$$

This is an integral of motion for an arbitrary Riemannian metric. It is preserved on an arbitrary geodesic

$$\frac{dE(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2} g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)\right) = 0 \,.$$

In fact we already know this integral of motion: Energy (5.22) is proportional to the square of the length of velocity vector:

$$|\mathbf{v}| = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2E} \,. \tag{5.23}$$

We already proved that velocity vector is preserved along the geodesic (see the Proposition in the subsection 3.2.1 and its proof (**??**).)

Example 2 Consider Riemannian metric $G = adu^2 + bdv^2$ (see also calculations in subsection 2.3.3) in the case if $a = a(u)$, $b = b(u)$, i.e. coefficients do not depend on the second coordinate v:

$$G = a(u)du^2 + b(u)dv^2, \; L_{\text{free}} = \frac{1}{2}\left(a(u)\dot{u}^2 + b(u)\dot{v}^2\right) \tag{5.24}$$

We see that Lagrangian does not depend on the second coordinate v hence the magnitude

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{v}} = b(u)\dot{v} \tag{5.25}$$

is preserved along geodesic. It is integral of motion because Euler-Lagrange equation for coordinate v is

$$\frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} - \frac{\partial L_{\text{free}}}{\partial v} = \frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} = \frac{d}{dt}F = 0 \;\; \text{since} \; \frac{\partial L_{\text{free}}}{\partial v} = 0. \,.$$

In fact all revolution surfaces which we consider here (cylinder, cone, sphere,...) have Riemannian metric of this type. Indeed consider further examples.

Example (sphere)

Sphere of the radius R in $\mathbf{E}^3$. Riemannian metric: $G = Rd\theta^2 + R^2\sin^2\theta d\varphi^2$ and $L_{\text{free}} = \frac{1}{2}\left(R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2\right)$ It is the case (5.24) for $u = \theta, v = \varphi$, $b(u) = R^2\sin^2\theta$ The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial\dot\varphi} = R^2\sin^2\theta\dot\varphi$$

Example (cone)

Consider cone $\begin{cases} x = ah\cos\varphi \\ y = ah\sin\varphi \\ z = bh \end{cases}$. Riemannian metric:

$$G = d(ah\cos\varphi)^2 + d(ah\sin\varphi)^2 + (dbh)^2 = (a^2+b^2)dh^2 + a^2h^2d\varphi^2\,.$$

and free Lagrangian

$$L_{\text{free}} = \frac{(a^2+b^2)\dot h^2 + a^2h^2\dot\varphi^2}{2}\,.$$

The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial\dot\varphi} = a^2h^2\dot\varphi.$$

Example (general surface of revolution)

Consider a surface of revolution in $\mathbf{E}^3$:

$$\mathbf{r}(h,\varphi)\colon\quad \begin{cases} x = f(h)\cos\varphi \\ y = f(h)\sin\varphi \\ z = h \end{cases} \qquad (f(h) > 0) \tag{5.26}$$

(In the case $f(h) = R$ it is cylinder, in the case $f(h) = kh$ it is a cone, in the case $f(h) = \sqrt{R^2 - h^2}$ it is a sphere, in the case $f(h) = \sqrt{R^2 + h^2}$ it is one-sheeted hyperboloid, in the case $z = \cosh h$ it is catenoid,...)

For the surface of revolution (5.26)

$$G = d(f(h)\cos\varphi)^2 + d(f(h)\sin\varphi)^2 + (dh)^2 = (f'(h)\cos\varphi dh - f(h)\sin\varphi d\varphi)^2 +$$
$$(f'(h)\sin\varphi dh + f(h)\cos\varphi d\varphi)^2 + dh^2 = (1 + f'^2(h))dh^2 + f^2(h)d\varphi^2\,.$$

The "free" Lagrangian of the surface of revolution is

$$L_{\text{free}} = \frac{\left(1 + f'^2(h)\right)\dot h^2 + f^2(h)\dot\varphi^2}{2}\,.$$

and the integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial\dot\varphi} = f^2(h)\dot\varphi.$$

5.3.6 Using integral of motions to calculate geodesics

Integrals of motions may be very useful to calculate geodesics. The equations for geodesics are second order differential equations. If we know integrals of motions they help us to solve these equations. Consider just an example.

For Lobachevsky plane the free Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$. We already calculated geodesics in the subsection 3.3.4. Geodesics are solutions of second order Euler-Lagrange equations for the Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ (see the subsection 3.3.4)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \end{cases}$$

It is not so easy to solve these differential equations.

For Lobachevsky plane we know two integrals of motions:

$$E = L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}, \quad \text{and } F = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}\,. \tag{5.27}$$

These both integrals preserve in time: if $x(t), y(t)$ is geodesics then

$$\begin{cases} F = \frac{\dot{x}(t)}{y(t)^2} \\ E = \frac{\dot{x}(t)^2+\dot{y}(t)^2}{2y(t)^2} = C_2 \end{cases} \Rightarrow \begin{cases} \dot{x} = C_1 y^2 \\ \dot{y} = \pm\sqrt{2C_2 y^2 - C_1^2 y^4} \end{cases}$$

These are first order differential equations. It is much easier to solve these equations in general case than initial second order differential equations.

5.3.7 Killing vectors of Lobachevsky plane and geodesics

Killing vector field of Rimeannina manifold (M, G) is an infinitesimal isometry of the Rimeannian metric G: under infinitesimal transform $x \to x + \varepsilon \mathbf{K}, x^i \to x^i + \varepsilon K^I(x)$ $(\varepsilon^2 = 0)$ metric does not change:

$$g_{ik}(x)dx^i dx^k = g_{ik}(x^r + \varepsilon K^i)(dx^i + \partial_m K^i dx^m)(dx^k + \partial_n K^k dx^n)\,. \tag{1}$$

Expanding this formula by ε and using the fact that $\varepsilon^2 = 0$ we come to

$$K^i \partial_i g_{km} + \partial_k K^r g_{rm} + \partial_m K^r g_{rk} = 0\,, \tag{1a}$$

(i.e. Lie derivative $\mathcal{L}_{\mathbf{K}} G = 0$.)

Examples: Killings of plane, sphere, cylindre, Lobachevsky plane.....

Theorem Let V be a vector space of all Killing vector fields of Riemannian manifold M. Then the dimension of V is less or equal than $\frac{n(n+1}{2)}$.

It means that for surfaces the number of independent Killing vector fields is less or equal to 3.

One can prove that it is only for plane, sphere and Lobachevsky plane that number of independent Killing vector fiels is equal to 3.

We calculate here Killing vector fields for Lobachevsky plane and use them for finding geodesics.

Theorem Let $\mathbf{K}$ be Killing tor field on Riemannian manifold (M, G) , and $L = \frac{g_{kp}\dot{x}^k\dot{x}^p}{2}$ Lagragian of 'free' particle on M. We know that geodesics are solutions of its equations of motions.

The magnitude

$$I = I_{\mathbf{K}} = K^i(x)\frac{\partial L(x,\dot{x})}{\partial \dot{x}^i}$$

is an integral of motion, i.e. it is preserved along geodesics.

The proof of the Theorem is obvious. The condition that $\mathbf{K}$ is Killing vector field means that

$$L(x^i + \varepsilon K^i, \dot{x}^i + \varepsilon \dot{K}^i)\,, \tag{2}$$

i.e.

$$K^i(x)\frac{\partial L}{\partial x^i} + \frac{dK^i}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0\,. \tag{2a}$$

Hence

$$\frac{d}{dt}I_{\mathbf{K}} = \frac{d}{dt}\left(K^i(x)\frac{\partial L(x,\dot{x})}{\partial \dot{x}^i}\right) = \frac{dK^i}{dt}\frac{\partial L(x,\dot{x})}{\partial \dot{x}^i} + K^i\frac{d}{dt}\left(\frac{\partial L(x,\dot{x})}{\partial \dot{x}^i}\right) =$$

$$\underbrace{K^i(x)\frac{\partial L}{\partial x^i} + \frac{dK^i}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{condition that } \mathbf{K} \text{ is Killing}} + K^i\underbrace{\left(\frac{\partial L(x,\dot{x})}{\partial x^i} - \frac{\partial L(x,\dot{x})}{\partial \dot{x}^i}\right)}_{\text{equations of motion}} = 0\,.$$

Use this Theorem to find geodesics.

First find Killing vector fields, i.e. infinitesimal isometries.

Since the dimension is equal 2, the dimension of space of Killing vector fields is ≤ 3. We will find three independent Killing vector fields.

There are two evident Killing vectors: Metric $G = \frac{dx^2+dy^2}{y^2}$ is evidently invariant with respect to translations $x \to x + a$ and homothety: $\begin{cases} x \to \lambda x \\ y \mapsto \lambda y \end{cases}$: $\frac{d(\lambda x)^2 + d(\lambda y)^2}{(\lambda y)^2} = \frac{dx^2+dy^2}{y^2}$.

Infinitesimal translation is $x' = x + \varepsilon, y' = y$, the vector field $D_1 = \frac{\partial}{\partial x}$. Infinitesimal homothety is $x' = x + \varepsilon x, y' = y + \varepsilon y$, the vector field $D_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Now most interesting: find the third Killing vector field. Use the fact that inversion $\mathbf{O}\colon (x,y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ preserves the metric. Consider the infinitesimal transformation $L_\varepsilon = \mathbf{O} \circ T_\varepsilon \circ \mathbf{O}$ ($L_0 = \mathbf{id}$):

$$L_\varepsilon\colon \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix} \to \begin{pmatrix} \frac{x}{x^2+y^2} + \varepsilon \\ \frac{y}{x^2+y^2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\frac{x}{x^2+y^2}+\varepsilon}{\left(\frac{x}{x^2+y^2}+\varepsilon\right)^2+\left(\frac{y}{x^2+y^2}\right)^2} \\ \frac{\frac{y}{x^2+y^2}}{\left(\frac{x}{x^2+y^2}+\varepsilon\right)^2+\left(\frac{y}{x^2+y^2}\right)^2} \end{pmatrix}.$$

6 Appendices to geometry on surfaces

6.1 Weingarten operator

6.1.1 Induced metric on surfaces.

Recall here again induced metric (see for detail subsection 1.4)

If surface $M\colon\ \mathbf{r} = \mathbf{r}(u,v)$is embedded in $\mathbf{E}^3$ then induced Riemannian metric G_M is defined by the formulae

$$\langle \mathbf{X}, \mathbf{Y} \rangle = G_M(\mathbf{X}, \mathbf{Y}) = G(\mathbf{X}, \mathbf{Y})\,, \tag{6.1}$$

where G is Euclidean metric in $\mathbf{E}^3$:

$$G_M = dx^2 + dy^2 + dz^2\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \sum_{i=1}^{3} (dx^i)^2\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \sum_{i=1}^{3} \left(\frac{\partial x^i}{\partial u^\alpha} du^\alpha\right)^2$$

$$= \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} du^\alpha du^\beta\,,$$

i.e.

$$G_M = g_{\alpha\beta} du^\alpha, \text{ where } g_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} du^\alpha du^\beta\,.$$

We use notations x, y, z or x^i ($i = 1,2,3$) for Cartesian coordinates in $\mathbf{E}^3$, u, v or u^α ($\alpha = 1,2$) for coordinates on the surface. We usually omit summation symbol over dummy indices. For coordinate tangent vectors

$$\underbrace{\frac{\partial}{\partial u_\alpha}}_{\text{Internal observer}} = \underbrace{\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}}_{\text{External observer}}$$

We have already plenty examples in the subsection 1.4. In particular for scalar product

$$g_{\alpha\beta} = \left\langle \frac{\partial}{u_\alpha}, \frac{\partial}{u_\beta} \right\rangle = x^i \alpha x^i \beta\,.\langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle\,. \tag{6.2}$$

6.1.2 Recalling Weingarten operator

Continue to play with formulae [8].

Recall the Weingarten (shape) operator which acts on tangent vectors:

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}\,, \tag{6.3}$$

where we denote by $\mathbf{n}$-unit normal vector field at the points of the surface M: $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 0$, $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 1$.

Now we realise that the derivative $\partial_{\mathbf{X}}\mathbf{R}$ of vector field with respect to another vector field is not a well-defined object: we need a connection. The formula $\partial_{\mathbf{X}}\mathbf{R}$ in Cartesian coordinates, is nothing but the derivative with respect to flat canonical connection: If we work only in Cartesian coordinates we do not need to distinguish between $\partial_{\mathbf{X}}\mathbf{R}$ and $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}$. Sometimes with some abuse of notations we will use $\partial_{\mathbf{X}}\mathbf{R}$ instead $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}$, but never forget: this can be done only in Cartesian coordinates where Christoffel symbols of flat canonical connection vanish:

$$\partial_{\mathbf{X}}\mathbf{R} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R} \quad \text{in Cartesian coordinates}\,.$$

So the rigorous definition of Weingarten operator is

$$S\mathbf{X} = -\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{n}\,, \tag{6.4}$$

but we often use the former one (equation (6.3)) just remembering that this can be done only in Cartesian coordinates.

Recall that the fact that Weingarten operator S maps tangent vectors to tangent vectors follows from the property: $\langle \mathbf{n}, \mathbf{X} \rangle = 0 \Rightarrow \mathbf{X}$ is tangent to the surface.

Indeed:

$$0 = \partial_{\mathbf{X}}\langle \mathbf{n}, \mathbf{n} \rangle = 2\langle \partial_{\mathbf{X}}\mathbf{n}, \mathbf{n} \rangle = -2\langle S\mathbf{X}, \mathbf{n} \rangle = 0 \Rightarrow S\mathbf{X} \text{ is tangent to the surface}$$

Recall also that normal unit vector is defined up to a sign, $\mathbf{n} \to -\mathbf{n}$. On the other hand if $\mathbf{n}$ is chosen then $\mathbf{S}$ is defined uniquely.

We use notations x, y, z or x^i ($i = 1, 2, 3$) for Cartesian coordinates in $\mathbf{E}^3$, u, v or u^α ($\alpha = 1, 2$) for coordinates on the surface. We usually omit summation symbol over dummy indices. For coordinate tangent vectors

$$\underbrace{\frac{\partial}{\partial u_\alpha}}_{\text{Internal observer}} \quad = \quad \underbrace{\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}}_{\text{External observer}}$$

[8] In some sense differential geometry it is when we write down the formulae expressing the geometrical facts, differentiate these formulae then reveal the geometrical meaning of the new obtained formulae e.t.c.

We have already plenty examples in the subsection 1.4. In particular for scalar product

$$g_{\alpha\beta} = \left\langle \frac{\partial}{u_\alpha}, \frac{\partial}{u_\beta} \right\rangle = x^i \alpha x^i \beta \, . \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle \, . \tag{6.5}$$

6.1.3 Second quadratic form

We define now the new object: *second quadratic form*

Definition. For two tangent vectors $\mathbf{X}, \mathbf{Y}$ $A(\mathbf{X}, \mathbf{Y})$ is defined such that

$$\left(\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y} \right)_\perp = A(\mathbf{X}, \mathbf{Y}) \mathbf{n} \tag{6.6}$$

i.e. we take orthogonal component of the derivative of $\mathbf{Y}$ with respect to $\mathbf{X}$.

This definition seems to be very vague: to evaluate covariant derivative we have to consider not a vector $\mathbf{Y}$ at a given point but the vector field. In fact one can see that $A(\mathbf{X}, \mathbf{Y})$ does depend only on the value of $\mathbf{Y}$ at the given point.

Indeed it follows from the definition of second quadratic form and from the properties of Weingarten operator that

$$A(\mathbf{X}, \mathbf{Y}) = \left\langle \left(\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y} \right)_\perp, \mathbf{n} \right\rangle = \left\langle \nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y}, \mathbf{n} \right\rangle =$$

$$\partial_{\mathbf{X}} \langle \mathbf{Y}, \mathbf{n} \rangle - \left\langle \mathbf{Y}, \nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{n} \right\rangle = \langle S(\mathbf{X}), \mathbf{Y} \rangle \tag{6.7}$$

We proved that second quadratic form depends only on value of vector field $\mathbf{Y}$ at the given poit and we established the relation between second quadratic form and Weingarten operator.

Proposition *The second quadratic form $A(\mathbf{X}, \mathbf{Y})$ is symmetric bilinear form on tangent vectors $\mathbf{X}, \mathbf{Y}$ in a given point.*

$$A \colon T_{\mathbf{p}} M \otimes T_{\mathbf{p}} M \to \mathbf{R}, \quad A(\mathbf{X}, \mathbf{Y}) = A(\mathbf{Y}, \mathbf{X}) = \langle S\mathbf{X}, \mathbf{Y} \rangle \, . \tag{6.8}$$

In components

$$A = A_{\alpha\beta} du^\alpha du^\beta = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n} \rangle = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} n^i \, . \tag{6.9}$$

and

$$S^\alpha_\beta = g^{\alpha\pi} A_{\pi\beta} = g^{\alpha\pi} x^i_{\pi\beta} n^i \, , \tag{6.10}$$

i.e.

$$A = GS, S = G^{-1} A \, .$$

Remark The normal unit vector field is defined up to a sign.

6.1.4 Gaussian and mean curvatures

Recall that Gaussian curvature

$$K = \det S$$

and mean curvature

$$H = \operatorname{Tr} S$$

It is easy to see that for Gaussian curvature

$$K = \det S = \det(G^{-1}A) = \frac{\det A}{\det G}.$$

We know already the geometrical meaning of Gaussian and mean curvatures from the point of view of the External Observer:

Gaussian curvature K equals to the product of principal curvatures, and mean curvartures equals to the sum of principal curvatures.

Now we ask a principal question: what bout internal observer, "aunt" living on the surface?

We will show that Gaussian curvature can be expressed in terms of induced Riemannian metric, i.e. it is an internal characteristic of the surface, invariant of isometries.

It is not the case with mean curvature: cylinder is isometric to the plane but it have non-zero mean curvature.

6.1.5 Examples of calculation of Weingarten operator, Second quadratic forms, curvatures for cylinder, cone and sphere.

Cylinder

We already calculated induced Riemannian metric on the cylinder (see (**??**)).

Cylinder is given by the equation $x^2+y^2=R^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}, \quad \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \qquad (6.11)$$

$$G_{cylinder} = \left(dx^2+dy^2+dz^2\right)\big|_{x=a\cos\varphi, y=a\sin\varphi, z=h} =$$

$$= (-a\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2\,, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}.$$

Normal unit vector $\mathbf{n} = \pm \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. Choose $\mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. Weingarten operator

$$S\partial_h = -\nabla^{\text{can.flat}}_{\mathbf{r}_h}\mathbf{n} = -\partial_{\mathbf{r}_h}\mathbf{n} = -\partial_h \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0\,,$$

$$S\partial_\varphi = -\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\partial_\varphi}{R}.$$

$$S\begin{pmatrix} \partial_h \\ \partial_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-\partial_\varphi}{R} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{R} \end{pmatrix}. \tag{6.12}$$

Calculate second quadratic form: $\mathbf{r}_{hh} = \partial_h \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} =$

$$\partial_h \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = 0,\ \mathbf{r}_{\varphi\varphi} = \partial_\varphi \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -R\cos\varphi \\ -R\sin\varphi \\ 0 \end{pmatrix} = -R\mathbf{n}\,.$$

We have

$$A_{\alpha\beta} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n} \rangle,\ A = \begin{pmatrix} \langle \mathbf{r}_{hh}, \mathbf{n} \rangle & \langle \mathbf{r}_{h\varphi}, \mathbf{n} \rangle \\ \langle \mathbf{r}_{\varphi h}, \mathbf{n} \rangle & \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -R \end{pmatrix}, \tag{6.13}$$

$$A = GS = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -R \end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = 0\,, \tag{6.14}$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = -\frac{1}{R}\,. \tag{6.15}$$

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

Cone

We already calculated induced Riemannian metric on the cone (see (**??**).

Cone is given by the equation $x^2+y^2-k^2z^2=0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h,\varphi)\colon\quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}, \quad \mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}, \tag{6.16}$$

$$G_{cone} = \left(dx^2+dy^2+dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} =$$
$$= (-kh\sin\varphi d\varphi + k\cos\varphi dh)^2 + (kh\cos\varphi d\varphi + k\sin\varphi dh)^2 + dh^2 =$$
$$k^2h^2d\varphi^2 + (k^2+1)dh^2, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix}.$$

One can see that $\mathbf{N} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$ is orthogonal to the surface: $\mathbf{N}\perp\mathbf{r}_h, \mathbf{r}_\varphi$. Hence normal unit vector $\mathbf{n} = \pm\frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$. Choose $\mathbf{n} = \frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$. Weingarten operator

$$S\partial_h = -\nabla^{\text{can.flat}}_{\mathbf{r}_h}\mathbf{n} = -\partial_{\mathbf{r}_h}\mathbf{n} = -\partial_h\left(\frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}\right) = 0\,,$$

$$S\partial_\varphi = -\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi\left(\frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}\right) =$$
$$\frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\partial_\varphi}{kh\sqrt{k^2+1}}.$$

$$S\begin{pmatrix} \partial_h \\ \partial_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\partial_\varphi}{kh\sqrt{k^2+1}} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix}. \tag{6.17}$$

Calculate second quadratic form: $\mathbf{r}_{hh} = \partial_h\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} =$

$$\partial_h\begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -k\sin\varphi \\ k\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{r}_{\varphi\varphi} = \partial_\varphi\begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix}.$$

We have

$$A_{\alpha\beta} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n}\rangle,\ A = \begin{pmatrix} \langle \mathbf{r}_{hh}, \mathbf{n}\rangle & \langle \mathbf{r}_{h\varphi}, \mathbf{n}\rangle \\ \langle \mathbf{r}_{\varphi h}, \mathbf{n}\rangle & \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{kh}{\sqrt{1+k^2}} \end{pmatrix}, \tag{6.18}$$

$$A = GS = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-kh}{\sqrt{k^2+1}} \end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix} = 0\,, \tag{6.19}$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix} = \frac{-1}{kh\sqrt{k^2+1}}\,. \tag{6.20}$$

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

Sphere

Sphere is given by the equation $x^2+y^2+z^2 = a^2$. Consider the parameterisation of sphere in spherical coordinates

$$\mathbf{r}(\theta,\varphi)\colon \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{6.21}$$

We already calculated induced Riemannian metric on the sphere (see (6.1.5)). Recall that

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$$

and

$$G_{S^2} = \left(dx^2+dy^2+dz^2\right)\big|_{x=R\sin\theta\cos\varphi, y=R\sin\theta\sin\varphi, z=R\cos\theta} =$$

$$(R\cos\theta\cos\varphi d\theta - R\sin\theta\sin\varphi d\varphi)^2 + (R\cos\theta\sin\varphi d\theta + R\sin\theta\cos\varphi d\varphi)^2 +$$

$$(-R\sin\theta d\theta)^2 = R^2\cos^2\theta d\theta^2 + R^2\sin^2\theta d\varphi^2 + R^2\sin^2\theta d\theta^2 =$$

$$= R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}.$$

For the sphere $\mathbf{r}(\theta,\varphi)$ is orthogonal to the surface. Hence normal unit vector $\mathbf{n}(\theta,\varphi)=\pm\frac{\mathbf{r}(\theta,\varphi)}{R}=\pm\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}$. Choose $\mathbf{n}=\frac{\mathbf{r}}{R}=\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}$. Weingarten operator

$$S\partial_\theta=-\nabla^{\text{can.flat}}_{\mathbf{r}_\theta}\mathbf{n}=-\partial_\theta\mathbf{n}=-\partial_\theta\left(\frac{\mathbf{r}}{R}\right)=-\frac{\mathbf{r}_\theta}{R}\,,$$
$$S\partial_\varphi=-\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n}=-\partial_\varphi\mathbf{n}=-\partial_\varphi\left(\frac{\mathbf{r}}{R}\right)=-\frac{\mathbf{r}_\varphi}{R}\,.$$
$$S\begin{pmatrix}\partial_\theta\\ \partial_\varphi\end{pmatrix}=\begin{pmatrix}-\frac{\partial_\theta}{R}\\ -\frac{\partial_\varphi}{R}\end{pmatrix},\quad S=-\begin{pmatrix}\frac{1}{R}&0\\ 0&\frac{1}{R}\end{pmatrix}\,. \tag{6.22}$$

For second quadratic form: $\mathbf{r}_{\theta\theta}=\partial_\theta\mathbf{r}_\theta=\begin{pmatrix}-R\sin\theta\cos\varphi\\ -R\sin\theta\sin\varphi\\ -R\cos\theta\end{pmatrix}$, $\mathbf{r}_{\theta\varphi}=\mathbf{r}_{\varphi\theta}=$

$$\partial_\theta\begin{pmatrix}-R\sin\theta\sin\varphi\\ R\sin\theta\cos\varphi\\ 0\end{pmatrix}=\begin{pmatrix}-R\cos\theta\sin\varphi\\ R\cos\theta\cos\varphi\\ 0\end{pmatrix},\ \mathbf{r}_{\varphi\varphi}=\partial_\varphi\begin{pmatrix}-R\sin\theta\sin\varphi\\ R\sin\theta\cos\varphi\\ 0\end{pmatrix}=\begin{pmatrix}-R\sin\theta\cos\varphi\\ -R\sin\theta\sin\varphi\\ 0\end{pmatrix}.$$

We have

$$A_{\alpha\beta}=\langle\mathbf{r}_{\alpha\beta},\mathbf{n}\rangle,\ A=\begin{pmatrix}\langle\mathbf{r}_{\theta\theta},\mathbf{n}\rangle&\langle\mathbf{r}_{\theta\varphi},\mathbf{n}\rangle\\ \langle\mathbf{r}_{\varphi\theta},\mathbf{n}\rangle&\langle\mathbf{r}_{\varphi\varphi},\mathbf{n}\rangle\end{pmatrix}=\begin{pmatrix}-R&0\\ 0&-R\sin^2\theta\end{pmatrix}, \tag{6.23}$$
$$A=GS=\begin{pmatrix}R^2&0\\ 0&R^2\sin^2\theta\end{pmatrix}\begin{pmatrix}\frac{-1}{R}&0\\ 0&\frac{-1}{R}\end{pmatrix}=-R\begin{pmatrix}1&0\\ 0&\sin^2\theta\end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K=\det S=\frac{\det A}{\det G}=\det\begin{pmatrix}-\frac{1}{R}&0\\ 0&-\frac{1}{R}\end{pmatrix}=\frac{1}{R^2}\,, \tag{6.24}$$

and mean curvature

$$H=\operatorname{Tr}S=\operatorname{Tr}\begin{pmatrix}-\frac{1}{R}&0\\ 0&-\frac{1}{R}\end{pmatrix}=-\frac{2}{R}\,, \tag{6.25}$$

Mean curvature is define up to a sign. If we change $\mathbf{n}\to-\mathbf{n}$ mean curvature $H\to\frac{1}{R}$ and Gaussian curvature will not change.

We see that for the sphere Gaussian curvature is not equal to zero, whilst for cylinder and cone Gaussian curvature equals to zero.

6.2 Derivation formula

Let M be a surface embedded in Euclidean space $\mathbf{E}^3$, $M\colon \mathbf{r} = \mathbf{r}(u,v)$.

Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface[9]. Vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ are functions on the surface M:

$$\mathbf{e} = \mathbf{e}(u,v),\ \ \mathbf{f} = \mathbf{f}(u,v)\,, \mathbf{n} = \mathbf{n}(u,v)\,.$$

Consider 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$:

$$d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u}du + \frac{\partial \mathbf{e}}{\partial v}dv,\ d\mathbf{f} = \frac{\partial \mathbf{f}}{\partial u}du + \frac{\partial \mathbf{f}}{\partial v}dv\,, d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial u}du + \frac{\partial \mathbf{n}}{\partial v}dv$$

These 1-forms take values in the vectors in $\mathbf{E}^3$, i.e. they are *vector valued* 1-forms. Any vector in $\mathbf{E}^3$ attached at an arbitrary point of the surface can be expanded over the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$. Thus vector valued 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ can be expanded in a sum of 1-forms with values in basic vectors $\mathbf{e}, \mathbf{f}, \mathbf{n}$. E.g. for $d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u}du + \frac{\partial \mathbf{e}}{\partial v}dv$ expanding vectors $\frac{\partial \mathbf{e}}{\partial u}$ and $\frac{\partial \mathbf{e}}{\partial v}$ over basis vectors we come to

$$\frac{\partial \mathbf{e}}{\partial u} = A_1(u,v)\mathbf{e} + B_1(u,v)\mathbf{f} + C_1(u,v)\mathbf{n},\ \frac{\partial \mathbf{e}}{\partial v} = A_2(u,v)\mathbf{e} + B_2(u,v)\mathbf{f} + C_2(u,v)\mathbf{n}$$

thus

$$\begin{aligned} d\mathbf{e} &= \frac{\partial \mathbf{e}}{\partial u}du + \frac{\partial \mathbf{e}}{\partial v}dv = (A_1\mathbf{e} + B_1\mathbf{f} + C_1\mathbf{n})\,du + (A_2\mathbf{e} + B_2\mathbf{f} + C_2\mathbf{n})\,dv = \\ &= \underbrace{(A_1du + A_2dv)}_{M_{11}}\mathbf{e} + \underbrace{(B_1du + B_2dv)}_{M_{12}}\mathbf{f} + \underbrace{(C_1du + C_2dv)}_{M_{11}}\mathbf{e}, \end{aligned} \tag{6.26}$$

i.e.

$$d\mathbf{e} = M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n},$$

where M_{11}, M_{12} and M_{13} are 1-forms on the surface M defined by the relation (6.26).

In the same way we do the expansions of vector-valued 1-forms $d\mathbf{f}$ and $d\mathbf{n}$ we come to

$$\begin{aligned} d\mathbf{e} &= M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n} \\ d\mathbf{f} &= M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n} \\ d\mathbf{n} &= M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n} \end{aligned}$$

[9]One can say that $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$ such that $\{\mathbf{e}, \mathbf{f}\}$ is an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$.

This equation can be rewritten in the following way:

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}=\begin{pmatrix}M_{11} & M_{12} & M_{13}\\ M_{21} & M_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} \tag{6.27}$$

Proposition The matrix M in the equation (6.27) *is antisymmetrical matrix, i.e.*

$$\begin{gathered}M_{11}=M_{22}=M_{33}=0\\ M_{12}=-M_{21}=a\\ M_{13}=-M_{31}=b\\ M_{23}=-M_{32}=-b\end{gathered} \tag{6.28}$$

i.e.

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}=\begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{6.29}$$

where a, b, c are 1*-forms on the surface M.*

Formulae (6.29) are called *derivation formula.*

Prove this Proposition. (Here I give the detailed proof, but later in Remark, very short proof in condensed notations.)

Recall that $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ is orthonormal basis, i.e. at every point of the surface

$$\langle\mathbf{e},\mathbf{e}\rangle=\langle\mathbf{f},\mathbf{f}\rangle=\langle\mathbf{n},\mathbf{n}\rangle=1,\text{ and }\langle\mathbf{e},\mathbf{f}\rangle=\langle\mathbf{e},\mathbf{n}\rangle=\langle\mathbf{f},\mathbf{n}\rangle=0$$

Now using (6.27) we have

$$\langle\mathbf{e},\mathbf{e}\rangle=1\Rightarrow d\langle\mathbf{e},\mathbf{e}\rangle=0=2\langle\mathbf{e},d\mathbf{e}\rangle=\langle\mathbf{e},M_{11}\mathbf{e}+M_{12}\mathbf{f}+M_{13}\mathbf{n}\rangle=$$

$$M_{11}\langle\mathbf{e},\mathbf{e}\rangle+M_{12}\langle\mathbf{e},\mathbf{f}\rangle+M_{13}\langle\mathbf{e},\mathbf{n}\rangle=M_{11}\Rightarrow M_{11}=0\,.$$

Analogously

$$\langle\mathbf{f},\mathbf{f}\rangle=1\Rightarrow d\langle\mathbf{f},\mathbf{f}\rangle=0=2\langle\mathbf{f},d\mathbf{f}\rangle=\langle\mathbf{f},M_{21}\mathbf{e}+M_{22}\mathbf{f}+M_{23}\mathbf{n}\rangle=M_{22}\Rightarrow M_{22}=0\,,$$

$$\langle\mathbf{n},\mathbf{n}\rangle=1\Rightarrow d\langle\mathbf{n},\mathbf{n}\rangle=0=2\langle\mathbf{n},d\mathbf{n}\rangle=\langle\mathbf{n},M_{31}\mathbf{e}+M_{32}\mathbf{f}+M_{33}\mathbf{n}\rangle=M_{33}.\Rightarrow M_{33}=0\,.$$

We proved already that $M_{11}=M_{22}=M_{33}=0$. Now prove that $M_{12}=-M_{21}$, $M_{13}=-M_{31}$ and $M_{13}=-M_{31}$.

$$\langle\mathbf{e},\mathbf{f}\rangle=0\Rightarrow d\langle\mathbf{e},\mathbf{f}\rangle=0=\langle\mathbf{e},d\mathbf{f}\rangle+\langle d\mathbf{e},\mathbf{f}\rangle=$$

$$\langle\mathbf{e},M_{21}\mathbf{e}+M_{22}\mathbf{f}+M_{23}\mathbf{n}\rangle+\langle M_{11}\mathbf{e}+M_{12}\mathbf{f}+M_{13}\mathbf{n},\mathbf{f}\rangle=M_{21}+M_{12}=0.$$

Analogously

$$\langle \mathbf{e}, \mathbf{n} \rangle = 0 \Rightarrow d\langle \mathbf{e}, \mathbf{n} \rangle = 0 = \langle \mathbf{e}, d\mathbf{n} \rangle + \langle d\mathbf{e}, \mathbf{n} \rangle =$$

$$\langle \mathbf{e}, M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n} \rangle + \langle M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n}, \mathbf{n} \rangle = M_{31} + M_{13} = 0$$

and

$$\langle \mathbf{f}, \mathbf{n} \rangle = 0 \Rightarrow d\langle \mathbf{f}, \mathbf{n} \rangle = 0 = \langle \mathbf{f}, d\mathbf{n} \rangle + \langle d\mathbf{f}, \mathbf{n} \rangle =$$

$$\langle \mathbf{f}, M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n} \rangle + \langle M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n}, \mathbf{n} \rangle = M_{32} + M_{23} = 0.$$

Remark This proof may be be performed much more shortly in condensed notations. Derivation formula (6.29) in condensed notations are

$$d\mathbf{e}_i = M_{ik}\mathbf{e}_k \tag{6.30}$$

Orthonormality condition means that $\langle \mathbf{e}_i, \mathbf{e}_k \rangle = \delta_{ik}$. Hence

$$d\langle \mathbf{e}_i, \mathbf{e}_k \rangle = 0 = \langle d\mathbf{e}_i, \mathbf{e}_k \rangle + \langle \mathbf{e}_i, d\mathbf{e}_k \rangle = \langle M_{im}\mathbf{e}_m, \mathbf{e}_k \rangle + \langle \mathbf{e}_i, M_{kn}\mathbf{e}_n \rangle = M_{ik} + M_{ki} = 0 \;\blacksquare \tag{6.31}$$

Much shorter, is not it?

6.2.1 Gauss condition (structure equations)

Derive the relations between 1-forms a, b and c in derivation formula.

Recall that a, b, c are 1-forms, $\mathbf{e}, f, \mathbf{n}$ are vector valued functions (0-forms) and $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ are vector valued 1-forms. (We use the simple identity that $ddf = 0$ and the fact that for 1-form $\omega \wedge \omega = 0$.) We have from derivation formula (6.29) that

$$d^2\mathbf{e} = 0 = d(a\mathbf{f} + b\mathbf{n}) = da\mathbf{f} - a \wedge d\mathbf{f} + db\mathbf{n} - b \wedge d\mathbf{n} =$$

$$da\,\mathbf{f} - a \wedge (-a\mathbf{e} + c\mathbf{n}) + db\,\mathbf{n} - b \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(da + b \wedge c)\mathbf{f} + (a \wedge a + b \wedge b)\mathbf{e} + (db - a \wedge c)\mathbf{n} = (da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0\,.$$

We see that

$$(da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0 \tag{6.32}$$

Hence components of the left hand side equal to zero:

$$(da + b \wedge c) = 0 \;\; (db + c \wedge a) = 0\,. \tag{6.33}$$

Analogously

$$d^2\mathbf{f} = 0 = d(-a\mathbf{e} + c\mathbf{n}) = -da\mathbf{e} + a \wedge d\mathbf{e} + dc\mathbf{n} - c \wedge d\mathbf{n} =$$

$$-da\mathbf{e} + a \wedge (a\mathbf{f} + b\mathbf{n}) + dc\,\mathbf{n} - c \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(-da + c \wedge b)\mathbf{e} + (dc + a \wedge b)\mathbf{n} = 0\,.$$

Hence we come to structure equations:

$$\begin{aligned} da + b \wedge c &= 0 \\ db + c \wedge a &= 0 \\ dc + a \wedge a &= 0 \end{aligned} \tag{6.34}$$

6.2.2 Geometrical meaning of derivation formula. Weingarten operator (shape oeprator) in terms of derivation formula.

Let M be a surface in $\mathbf{E}^3$.

Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Note that in generally these vectors are not coordinate vectors.

Describe Riemannian geometry on the surface M in terms of this basis and derivation formula (6.29).

Induced Riemannian metric

If G is the Riemannian metric induced on the surface M then since $\mathbf{e}, \mathbf{f}$ is orthonormal basis at every tangent space $T_{\mathbf{p}}M$ then

$$G(\mathbf{e}, \mathbf{e}) = G(\mathbf{f}, \mathbf{f}) = 1, \;\; G(\mathbf{e}, \mathbf{f}) = G(\mathbf{f}, \mathbf{e}) = 0 \tag{6.35}$$

The matrix of the Riemannian metric in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.36}$$

Induced connection Let ∇ be the connection induced by the canonical flat connection on the surface M.

Then according equations (**??**) and derivation formula (6.29) for every tangent vector $\mathbf{X}$

$$\nabla_{\mathbf{X}}\mathbf{e} = (\partial_{\mathbf{X}}\mathbf{e})_{\text{tangent}} = (d\mathbf{e}(\mathbf{X}))_{\text{tangent}} = (a(\mathbf{X})\mathbf{f} + b(\mathbf{X})\mathbf{n})_{\text{tangent}} = a(\mathbf{X})\mathbf{f}\,. \tag{6.37}$$

and

$$\nabla_{\mathbf{X}}\mathbf{f} = (\partial_{\mathbf{X}}\mathbf{f})_{\text{tangent}} = (d\mathbf{f}(\mathbf{X}))_{\text{tangent}} = (-a(\mathbf{X})\mathbf{e} + c(\mathbf{X})\mathbf{n})_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}\,. \tag{6.38}$$

In particular

$$\begin{matrix} \nabla_{\mathbf{e}}\mathbf{e} = a(\mathbf{e})\mathbf{f} & \nabla_{\mathbf{f}}\mathbf{e} = a(\mathbf{f})\mathbf{f} \\ \nabla_{\mathbf{e}}\mathbf{f} = -a(\mathbf{e})\mathbf{e} & \nabla_{\mathbf{f}}\mathbf{f} = -a(\mathbf{f})\mathbf{e} \end{matrix} \tag{6.39}$$

We know that the connection ∇ is Levi-Civita connection of the induced Riemannian metric (6.37) (see the subsection 4.2.1)[10].

Second Quadratic form Second quadratic form is a bilinear symmetric function $A(\mathbf{X},\mathbf{Y})$ on tangent vectors which is well-defined by the condition $A(\mathbf{X},\mathbf{Y})\mathbf{n} = (\partial_{\mathbf{X}}\mathbf{Y})_{\text{orthogonal}}$ (see e.g. subsection 6.4 in Appendices.)

Let $A(\mathbf{X},\mathbf{Y})$ be second quadratic form. Then according to derivation formula (6.29) we have

$$A(\mathbf{e},\mathbf{e}) = \langle \partial_{\mathbf{e}}\mathbf{e},\mathbf{n}\rangle = \langle d\mathbf{e}(\mathbf{e}),\mathbf{n}\rangle = \langle a(\mathbf{e})\mathbf{f} + b(\mathbf{e})\mathbf{n},\mathbf{n}\rangle = b(\mathbf{e})\,,$$
$$A(\mathbf{f},\mathbf{e}) = \langle \partial_{\mathbf{f}}\mathbf{e},\mathbf{n}\rangle = \langle d\mathbf{e}(\mathbf{f}),\mathbf{n}\rangle = \langle a(\mathbf{f})\mathbf{f} + b(\mathbf{f})\mathbf{n},\mathbf{n}\rangle = b(\mathbf{f})\,,$$
$$A(\mathbf{e},\mathbf{f}) = \langle \partial_{\mathbf{e}}\mathbf{f},\mathbf{n}\rangle = \langle d\mathbf{f}(\mathbf{e}),\mathbf{n}\rangle = \langle -a(\mathbf{e})\mathbf{f} + c(\mathbf{e})\mathbf{n},\mathbf{n}\rangle = c(\mathbf{e})\,,$$
$$A(\mathbf{f},\mathbf{f}) = \langle \partial_{\mathbf{f}}\mathbf{f},\mathbf{n}\rangle = \langle d\mathbf{f}(\mathbf{f}),\mathbf{n}\rangle = \langle -a(\mathbf{f})\mathbf{f} + c(\mathbf{f})\mathbf{n},\mathbf{n}\rangle = c(\mathbf{f})\,,$$

The matrix of the second quadratic form in the basis $\{\mathbf{e},\mathbf{f}\}$ is

$$A = \begin{pmatrix} A(\mathbf{e},\mathbf{e}) & A(\mathbf{f},\mathbf{e}) \\ A(\mathbf{e},\mathbf{f}) & A(\mathbf{f},\mathbf{f}) \end{pmatrix} = \begin{pmatrix} b(\mathbf{e}) & b(\mathbf{f}) \\ c(\mathbf{e}) & c(\mathbf{f}) \end{pmatrix} \tag{6.41}$$

This is symmetrical matrix (see the subsection 4.3.2):

$$A(\mathbf{f},\mathbf{e}) = b(\mathbf{f}) = A(\mathbf{e},\mathbf{f}) = c(\mathbf{e})\,. \tag{6.42}$$

Weingarten (Shape) operator

Let S be Weingarten operator: $S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}$ (see the subsection 6.4 in Appendix, or Geometry lectures). Then it follows from derivation formula that

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n} = -d\mathbf{n}(\mathbf{X}) = -\left(-b(X)\mathbf{e} - c(\mathbf{X})\mathbf{f}\right) = b(\mathbf{X})\mathbf{e} + c(\mathbf{X})f$$

In particular

$$S(\mathbf{e}) = b(\mathbf{e})\mathbf{e} + c(\mathbf{e})\mathbf{f}\,, S(\mathbf{f}) = b(\mathbf{f})\mathbf{e} + c(\mathbf{f})\mathbf{f}$$

and the matrix of the Weingarten operator in the basis $\{\mathbf{e},\mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & b(\mathbf{f}) \\ c(\mathbf{e}) & c(\mathbf{f}) \end{pmatrix} \tag{6.43}$$

Remark According to the condition (6.42) the matrix S is symmetrical. The relations $A = GS, S = G^{-1}A$ for Weingarten operator, Riemannian metric and second quadratic form are evidently obeyed for matrices of these operators in the basis $\mathbf{e},\mathbf{f}$ where $G = 1$, $A = S$.

[10] In particular this implies that this is symmetric connection, i.e.

$$\nabla \mathbf{f}\mathbf{e} - \nabla_{\mathbf{e}}\mathbf{f} - [\mathbf{f},\mathbf{e}] = a(\mathbf{f})\mathbf{f} + a(\mathbf{e})\mathbf{e} - [\mathbf{f},\mathbf{e}] = 0\,. \tag{6.40}$$

6.2.3 Gaussian and mean curvature in terms of derivation formula

Now we are equipped to express Gaussian and mean curvatures in terms of derivation formula. Using (6.43) we have for Gaussian curvature

$$K = \det S = b(\mathbf{e})c(\mathbf{f}) - c(\mathbf{e})b(\mathbf{f}) = (b \wedge c)(\mathbf{e}, \mathbf{f}) \tag{6.44}$$

and for mean curvature

$$H = \mathrm{Tr}\, S = b(\mathbf{e}) + c(\mathbf{f}) \tag{6.45}$$

What next? We will study in more detail formula (6.44) later.

Now consider some examples of calculation of Weingarten operator, e.t..c. for using derivation formula.

6.2.4 Calculations with use of derivation formulae

6.2.5 Examples of calculations of derivation formulae and curvatures for cylinder, cone and sphere

Last year we calculated Weingarten operator and curvatures for cylinder, cone and sphere (see also the subsection 6.4 in Appendices.). Now we do the same but in terms of derivation formula.

Cylinder

We have to define three vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ on the points of the cylinder surface $x^2 + y^2 = a^2$:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} \tag{6.46}$$

such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. We calculated many times coordinate vector fields $\mathbf{r}_h, \mathbf{r}_\varphi$ and normal unit vector field:

$$\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}. \tag{6.47}$$

Vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ and $\mathbf{n}$ are orthogonal to each other but not all of them have unit length. One can choose

$$\mathbf{e} = \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{f} = \frac{\mathbf{r}_\varphi}{a} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{6.48}$$

These vectors form an orthonormal basis and $\mathbf{e}, \mathbf{f}$ form an orthonormal basis in tangent space.

Derive for this basis derivation formula (6.29). For vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ in (6.48) we have

$$d\mathbf{e} = 0, d\mathbf{f} = d\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} d\varphi = -\mathbf{n}d\varphi,$$

$$d\mathbf{n} = d\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \mathbf{f}d\varphi,$$

i.e.

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -d\varphi \\ 0 & d\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \tag{6.49}$$

i.e. in derivation formula 1-forms a, b vanish $a = b = 0$ and $c = -d\varphi$.

The matrix of Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} 0 & -d\varphi(\mathbf{e}) \\ 0 & -d\varphi(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$$

According to (6.44) and(6.45) Gaussian curvature $K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = 0$ and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -d\varphi(\mathbf{f}) = -d\varphi\left(\frac{\mathbf{r}_\varphi}{R}\right) = -\frac{1}{a}$$

Remark We denote by the same letter a the radius of the cylinder surface (6.46) and 1-form a in derivation formula. I hope that this will not lead to the confusion. (May be it is better to denote the radius of the cylindrical surface by the letter R.)

Cone

For cone:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases},$$

$$\mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

Tangent vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ are orthogonal to each other. The length of the vector $\mathbf{r}_h$ equals to $\sqrt{1+k^2}$ and the length of the vector $\mathbf{r}_\varphi$ equals to kh. Hence we can choose orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ such that vectors $\mathbf{e}, \mathbf{f}$ are unit vectors in the directions of the vectors $\mathbf{r}_h, \mathbf{r}_\varphi$:

$$\mathbf{e} = \frac{\mathbf{r}_h}{\sqrt{1+k^2}} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \ \mathbf{f} = \frac{\mathbf{r}_\varphi}{hk} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

Calculate $d\mathbf{e}, d\mathbf{f}$ and $d\mathbf{n}$:

$$d\mathbf{e} = d\begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix} = \frac{kd\varphi}{\sqrt{1+k^2}} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \frac{kd\varphi}{\sqrt{1+k^2}}\mathbf{f},$$

$$d\mathbf{f} = d\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} d\varphi =$$

$$\frac{-k}{1+k^2}\begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix} d\varphi - \frac{d\varphi}{1+k^2}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = \frac{-kd\varphi}{\sqrt{1+k^2}}\mathbf{e} - \frac{d\varphi}{\sqrt{1+k^2}}\mathbf{n},$$

and

$$d\mathbf{n} = \frac{1}{\sqrt{1+k^2}} d\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = \frac{d\varphi}{\sqrt{1+k^2}}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}.$$

We come to

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \frac{kd\varphi}{\sqrt{1+k^2}} & 0 \\ -\frac{kd\varphi}{\sqrt{1+k^2}} & 0 & \frac{-d\varphi}{\sqrt{1+k^2}} \\ 0 & \frac{d\varphi}{\sqrt{1+k^2}} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \tag{6.50}$$

i.e. in derivation formula for 1-forms $a = \frac{kd\varphi}{\sqrt{1+k^2}}$, $b = 0$ and and $c = -\frac{-d\varphi}{\sqrt{1+k^2}}$.

Remark Note that calculation of $d\mathbf{f}$ are little bit hard. On the other hand the answer for $d\mathbf{f}$ follows from answers for $d\mathbf{e}$ and $d\mathbf{n}$ since the matrix in (6.50) is antisymmetric. So we can omit the straightforward calculations of $d\mathbf{f}$.

The matrix of Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = S = \begin{pmatrix} 0 & \frac{-d\varphi(\mathbf{e})}{\sqrt{1+k^2}} \\ 0 & \frac{-d\varphi(\mathbf{f})}{\sqrt{1+k^2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{1+k^2}} \end{pmatrix}.$$

since $d\varphi(\mathbf{f}) = d\varphi\left(\frac{\mathbf{r}_\varphi}{kh}\right) = \frac{1}{kh}d\varphi(\partial_\varphi) = \frac{1}{kh}$.

According to (6.44), (6.45) Gaussian curvature

$$K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = 0$$

and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -d\varphi(\mathbf{f}) = -d\varphi\left(\frac{\mathbf{r}_\varphi}{R}\right) = -\frac{1}{kh\sqrt{1+k^2}}\,.$$

Sphere

For sphere

$$\mathbf{r}(\theta,\varphi)\colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{6.51}$$

$$\mathbf{r}_\theta(\theta,\varphi) = \frac{\partial \mathbf{r}}{\partial\theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \quad \mathbf{r}_\varphi(\theta,\varphi) = \frac{\partial \mathbf{r}}{\partial\varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix},$$

$$\mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

Tangent vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$ are orthogonal to each other. The length of the vector $\mathbf{r}_\theta$ equals to R and the length of the vector $\mathbf{r}_\varphi$ equals to $R\sin\theta$. Hence we can choose orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ such that vectors $\mathbf{e},\mathbf{f}$ are unit vectors in the directions of the vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$:

$$\mathbf{e}(\theta,\varphi) = \frac{\mathbf{r}_\theta}{R} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}, \mathbf{f}(\theta,\varphi) = \frac{\mathbf{r}_\varphi}{R\sin\theta} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

Calculate $d\mathbf{e}, d\mathbf{f}$ and $d\mathbf{n}$:

$$d\mathbf{e} = d\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} =$$

$$\begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} d\varphi - \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix} d\theta = \cos\theta d\varphi\mathbf{f} - d\theta\mathbf{n},$$

$$d\mathbf{f} = d\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = -\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} d\varphi =$$

$$-\cos\theta d\varphi \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} - \sin\theta d\varphi \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = -\cos\theta d\varphi\mathbf{e} - \sin\theta d\varphi\mathbf{n}\,,$$

$$d\mathbf{n} = d\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix} = \begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix} d\theta + \begin{pmatrix}-\sin\theta\sin\varphi\\ \sin\theta\cos\varphi\\ 0\end{pmatrix} d\varphi$$
$$= d\theta\mathbf{e} + \sin\theta d\varphi\mathbf{f}\,.$$

i.e.

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & \cos\theta d\varphi & -d\theta\\ -\cos\theta d\varphi & 0 & -\sin\theta d\varphi\\ d\theta & \sin\theta d\varphi & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{6.52}$$

i.e. in derivation formula $a = \cos\theta d\varphi$, $b = -d\theta$, $c = -\sin\theta d\varphi$.

Remark The same remark as for cone: equipped by the properties of derivation formula we do not need to calculate $d\mathbf{f}$. The calculation of $d\mathbf{e}$ and $d\mathbf{n}$ and the property that the matrix in derivation formula is antisymmetric gives us the answer for $d\mathbf{f}$.

The matrix of Weingarten operator in the basis $\{\mathbf{e},\mathbf{f}\}$ is

$$S = \begin{pmatrix}b(\mathbf{e}) & c(\mathbf{e})\\ b(\mathbf{f}) & c(\mathbf{f})\end{pmatrix} = \begin{pmatrix}-d\theta(\mathbf{e}) & -\sin\theta d\varphi(\mathbf{e})\\ -d\theta(\mathbf{f}) & -\sin\theta d\varphi(\mathbf{f})\end{pmatrix} = \begin{pmatrix}\frac{-1}{R} & 0\\ 0 & -\frac{1}{R}\end{pmatrix}$$

since $d\theta(\mathbf{e}) = d\theta\left(\frac{\partial_\theta}{R}\right) = \frac{1}{R}d\theta(\partial_\theta) = \frac{1}{R}$, $d\varphi(\mathbf{e}) = d\varphi\left(\frac{\partial_\theta}{R}\right) = \frac{1}{R}d\varphi(\partial_\theta) = 0$.

According to (6.44) and(6.45) Gaussian curvature

$$K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = \frac{1}{R^2}$$

and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -\frac{2}{R}$$

Notice that for calculation of Weingarten operator and curvatures we used only 1-forms b and c, i.e. the derivation equation for $d\mathbf{n}$, ($d\mathbf{n} = d\theta\mathbf{e} + \sin\theta d\varphi\mathbf{f}$).

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

We see that for the sphere Gaussian curvature is not equal to zero, whilst for cylinder and cone Gaussian curvature equals to zero.

6.2.6 Proof of the Theorem of parallel transport along closed curve

We are ready now to prove the Theorem. Recall that the Theorem states following:

If C is a closed curve on a surface M such that C is a boundary of a compact oriented domain $D \subset M$, then during the parallel transport of an arbitrary tangent vector along the closed curve C the vector rotates through the angle

$$\Delta\Phi = \angle(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = \int_D K d\sigma\,, \tag{6.53}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g} dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g} dudv$.

(see (**??**).

Recall that for derivation formula (6.29) we obtained structure equations

$$\begin{aligned} da + b \wedge c = 0 \\ db + c \wedge a = 0 \\ dc + a \wedge a = 0 \end{aligned} \tag{6.54}$$

We need to use only one of these equations, the equation

$$da + b \wedge c = 0\,. \tag{6.55}$$

This condition sometimes is called *Gauß condition.*

Let as always $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ be an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$ such that $\{\mathbf{e}, \mathbf{f}\}$ is an orthonormal basis in $T_{\mathbf{p}}M$ at every point of surface $\mathbf{p} \in M$. Then the Gauß condition (6.72) and equation (6.44) mean that for Gaussian curvature on the surface M can be expressed through the 2-form da and base vectors $\{\mathbf{e}, \mathbf{f}\}$:

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) \tag{6.56}$$

We use this formula to prove the Theorem.

Now calculate the parallel transport of an arbitrary tangent vector over the closed curve C on the surface M.

Let $\mathbf{r} = \mathbf{r}(u, v) = \mathbf{r}(u\alpha)$ $(\alpha = 1, 2,\ (u, v) = (u^1, v^1))$ be an equation of the surface M.

Let $u^\alpha = u^\alpha(t)$ $(\alpha = 1, 2)$ be the equation of the curve C. Let $\mathbf{X}(t)$ be the parallel transport of vector field along the closed curve C, i.e. $\mathbf{X}(t)$ is tangent to the surface M at the point $u(t)$ of the curve C and vector field $\mathbf{X}(t)$ is covariantly constant along the curve:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0$$

To write this equation in components we usually expanded the vector field in the coordinate basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and used Christoffel symbols of the connection $\Gamma^\alpha_{\beta\gamma}$: $\nabla_\beta \partial_\gamma = \Gamma^\alpha_{\beta\gamma}\partial_\alpha$.

Now we will do it in different way: *instead coordinate basis* $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ *we will use the basis* $\{\mathbf{e}, \mathbf{f}\}$. In the subsection 3.4.4 we obtained that the connection ∇ has the following appearance in this basis

$$\nabla_{\mathbf{v}}\mathbf{e} = a(\mathbf{v})\mathbf{f},\ , \nabla_{\mathbf{v}}\mathbf{f} = -a(\mathbf{v})\mathbf{e} \tag{6.57}$$

(see the equations (6.37) and (6.38))

Let

$$\mathbf{X} = \mathbf{X}(u(t)) = X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t))$$

Lbe an expansion of tangent vector field $\mathbf{X}(t)$ over basis $\{\mathbf{e}, \mathbf{f}\}$. Let $\mathbf{v}$ be velocity vector of the curve C. Then the equation of parallel transport $\frac{\nabla \mathbf{X}(t)}{dt} = 0$ will have the following appearance:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0 = \nabla_{\mathbf{v}}\left(X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t))\right) =$$

$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)\nabla_{\mathbf{v}}\mathbf{e}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) + X^2(t)\nabla_{\mathbf{v}}\mathbf{f}(u(t)) =$$

$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)a(\mathbf{v})\mathbf{f}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) - X^2(t)a(\mathbf{v})\mathbf{e}(u(t)) =$$

$$\left(\frac{dX^1(t)}{dt} - X^2(t)a(\mathbf{v})\right)\mathbf{e}(u(t)) + \left(\frac{dX^2(t)}{dt} + X^1(t)a(\mathbf{v})\right)\mathbf{f}(u(t)) = 0.$$

Thus we come to equation:

$$\begin{cases} \dot{X}^1(t) - a(\mathbf{v}(t))X^2 = 0 \\ \dot{X}^2(t) + a(\mathbf{v}(t))X^1 = 0 \end{cases}$$

There are many ways to solve this equation. It is very convenient to consider complex variable

$$Z(t) = X^1(t) + iX^2(t)$$

We see that

$$\dot{Z}(t) = \dot{X}^1(t) + i\dot{X}^2(t) = a(\mathbf{v}(t))X^2 - ia(\mathbf{v}(t)X^1 = -ia(\mathbf{v})Z(t),$$

i.e.

$$\frac{dZ(t)}{dt} = -ia(\mathbf{v}(t))Z(t) \tag{6.58}$$

The solution of this equation is:

$$\tag{6.59}$$

Calculate $\int_0^{t_1} a(\mathbf{v}(\tau))d\tau$ for closed curve $u(0) = u(t_1)$. Due to Stokes Theorem:

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da$$

Hence using Gauss condition (6.72) we see that

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da = -\int_D b \wedge c$$

Claim

$$\int_D b \wedge c = -\int_D da = \int K d\sigma\,. \tag{6.60}$$

Theorem follows from this claim:

$$Z(t_1) = Z(0)e^{-i\int_C a} = Z(0)e^{i\int_D b\wedge C} \tag{6.61}$$

Denote the integral $i\int_D b \wedge C$ by $\Delta\Phi$: $\Delta\Phi = i\int_D b \wedge C$. We have

$$Z(t_1) = X^1(t_1) + iX^2(t_1) = \left(X^1(0) + iX^2(0)\right) e^{i\Delta\Phi} = \tag{6.62}$$

It remains to prove the claim. The induced volume form $d\sigma$ is 2-form. Its value on two orthogonal unit vector $\mathbf{e}, \mathbf{f}$ equals to 1:

$$d\sigma(\mathbf{e}, \mathbf{f}) = 1 \tag{6.63}$$

(In coordinates u, v volume form $d\sigma = \sqrt{\det g} du \wedge dv$).

The value of the form $b \wedge c$ on vectors $\{\mathbf{e}, \mathbf{f}\}$ equals to Gaussian curvature according to (6.73). We see that

$$b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) = K d\sigma(\mathbf{e}, \mathbf{f})$$

Hence 2-forms $b \wedge c$, $-da$ and volume form $d\sigma$ coincide. Thus we prove (6.77).

6.3 Formula for Gaussian curvature in isothermal (conformal) coordinates

Now we will consider one very beautiful and illuminating formula to calculate Gaussian curvature for surfaces.

Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$.

Definition We say that coordinates (parameters) u, v are *isothermal* (or *conformal*) if the induced Riemannian metric **??** is equal to

$$G = \sigma(u, v)(du^2 + dv^2)$$

Consider examples. For example if (u, v) are stereographic coordinates for sphere of radius R we know that

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2)}$$

i.e. stereographic coordiantes are conformal coordiantes:

$$G = \sigma(u, v)(du^2 + dv^2) \text{ with } \sigma = \frac{4R^4}{(R^2 + u^2 + v^2)^2}\,.$$

Another example: an arbitrary locally Euclidean surface, i.e. surface with induced Riemannian metric $G = du^2 + dv^2$ in some local coordinates u, v.

One can show that locally one can always find conformal (isothermal) coordinates on surface in $\mathbf{E}^3$ [11]

Theorem Let surface M is given in conformal coordinates: $\mathbf{r} = \mathbf{r}(u, v)$ such that induced Riemannian metric is equal to $G = \sigma(u, v)(du^2+dv^2)$. Then Gaussian curvature K of the surface M is given by the formula

$$K = -\frac{1}{2}\frac{\Delta(\log\sigma)}{\sigma}\,, \qquad \text{where } \Delta = \tfrac{\partial^2}{\partial u^2} + \tfrac{\partial^2}{\partial v^2}\,. \tag{6.64}$$

In particular this formula states that Gaussian curvature is expressed in terms of Riemannian metric. This implies Gauß *Theorema Egregium.*

If u, v are conformal coordinates, $G = \sigma(du^2 + dv^2)$, then it is often convenient to introduce a function $\Phi(u, v)$ such that $\Phi = \log\sigma$, i.e. $\sigma = e^{\Phi(u,v)}$. Then

$$G = e^{\Phi(u,v)}(du^2 + dv^2)\,. \tag{6.65}$$

Then formula (6.64) will have the following appearance:

$$K = -\frac{1}{2}e^{-\Phi}\Delta(\Phi) = -\frac{1}{2}e^{-\Phi}\left(\frac{\partial^2\Phi(u, v)}{\partial u^2} + \frac{\partial^2\Phi(u, v)}{\partial v^2}\right)\,. \tag{6.66}$$

What about existence of conformal (isothermal) coordinates? **Proposition** For surface M in $\mathbf{E}^3$

- in a vicinity of an arbitrary point there exist isothermal coordinates i.e. coordinates such that induced metric $G = e^{\Phi}(du^2 + dv^2) = e^{\Phi}dzd\bar{z}$.
- If (u, v) and (u', v') are two arbitrary isothermal coordinates then the function $z = f(w)$ is holomorphic function or anti-holomorphic function,

[11] The existence of local isothermal coordiantes is a part of famous Gauss theorem, which can be formulated in modern terms in the following way: every surface has a canonical complex structure ($z = u + iv, \bar{z} = u - iv$). We will consider this question later.

We denote $u+iv=z, u-iv=\bar{z}$. Recall that if $z=u+iv$ then

$$F_z=\frac{\partial F}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\right)F,\quad \textit{and } F_{\bar{z}}=\frac{\partial F}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)F \tag{6.67}$$

Function $F=f+ig$ is holomorphic $\Leftrightarrow F_{\bar{z}}=0 \Leftrightarrow (f_u+ig_u)+i(f_v+ig_v)=0 \Leftrightarrow f_u=g_v$ and $f_v=-g_v$ (Cauchy Riemann conditions). Function $F=f+ig$ is anti-holomorphic $\Leftrightarrow F_z=0 \Leftrightarrow (f_u+ig_u)-i(f_v-ig_v)=0$. E.g. $F=z^2=(u+iv)^2=u^2-v^2+2iuv$ is holomorphic function, $F=e^{\bar{z}}=e^{u-iv}=e^u(\cos v-i\sin v)$ is anti-holomorphic function (See also **??**).

This Proposition immediately implies the following important corollary-Theorem:

Two-dimensional surface in $\mathbf{E}^3$ *has canonical complex structure, i.e. one can consider a canonical atlas of local complex charts such that transition funtctions are analytic.*

Idea of Proof of this Proposition

Let (ξ,η) be arbitrary parameters of surface and $G=Ad\xi^2+2Bd\xi d\eta+Cd\eta^2$ ($g_{11}=A, g_{12}=g_{21}=B$, $g_{22}=C$). The positive-definiteness of the metric implies that $G=\omega\overline{\omega}$ where $\omega=df+idg$ is 1-form. Use the fact that an arbitrary 1-form up to a mulitplier function is an exact form: $\omega=\lambda dF$. We come to isothermal coordinates: $G=\lambda\bar{\lambda}dFd\bar{F}$. To prove the second part of Proposition we just perform straightforward calculation. Let $G=e^{\Phi}(du^2+dv^2)=e^{\Phi}dzd\bar{z}$ in local coordinates $z=u+iv$, and in new local coordinates $w=u'+iv'$ $G=e^{\Phi'}(du'^2+dv'^2)=e^{\Phi'}dwd\bar{w}$, where Let $w=F(z)$. Then

$$G=e^{\Phi}dzd\bar{z}=e^{\Phi}\left(F_w dw+F_{\bar{w}}d\bar{w}\right)\left(\overline{F_w dw+F_{\bar{w}}d\bar{w}}\right)=$$

$$e^{\Phi}\left(F_w\overline{F_{\bar{w}}}dw^2+\left(|F_w|^2+|F_{\bar{w}}|^2|\right)dwd\bar{w}+F_{\bar{w}}\overline{F_w}d\bar{w}^2\right) \tag{6.68}$$

The condition that new coordinates are isothermal too means that $F_w\overline{F_{\bar{w}}}=0$, i.e. $F_w=0$, i.e. F is anti-holomorphic function or $F_{\bar{w}}=0$, i.e. F is holomorphic function.

Illustrate this idea on the example: Let $G=dx^2+f^2(x)dy^2$ be a metric on a domain of Riemannian manifold (e.g. for sphere $x=\theta, y=\varphi, f(x)=\sin^2 x$, for cone $x=h, y=\varphi, f(x)=x$). Then $G=(dx+ifdy)()dx-if(x)dy)$. For 1-form $\omega=dx+if(x)dy$ we have that $dx+if(x)dy=f(x)\left(dG(x)+idy\right)=f(x)d\left(L(x)+iy\right)$, where $L(x)$ is antiderivative of a function $\frac{1}{f}$ and $dx^2+f^2(x)dy^2=f^2(x)d\left(L(x)+iy\right)d\left(G(x)-iy\right)=e^{\Phi}(du^2+dv^2)$, where $e^{\Phi}=f^2(x)$, $u=L(x), v=y$ [12]

[12] in general case we use essentially the condition of analiticity. This proof was done by Gauss. The general smooth case was proved only in the beginning of XX century.

6.4 Curvatures for surface $z = F(x, y)$

.

Now using derivation formulae we calculate curvature for arbitrary surface $z = F(x, y)$ and later we will calculate curvatures for surfaces in conformal (isometric coordinates).

We will caclulate curvature not at an arbitrary point but only at the points of extrema of function $F(x, y)$. (In fact this condition is not very demanding.)

Derivation formulae become very useful tool for solving these questions [13].

A surface $z = F(x, y)$ can be defined by parameterisation

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases}$$

Conisder coordinate vector fields of the surface

$$\frac{\partial}{\partial u} - \mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix}, \frac{\partial}{\partial v} - \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix},$$

and unit normal vector field

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}.$$

It is obviously orthogonal to $\mathbf{r}_u, \mathbf{r}_v$ and it has unbit length.

One can see that $\mathbf{e} = \frac{\mathbf{r}_u}{|\mathbf{r}_u|} = \frac{1}{\sqrt{1+F_u^2}} \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix}$ is unit vector field tangent to surface. The vector field $\frac{\mathbf{r}_v}{|\mathbf{r}_v|} = \frac{1}{\sqrt{1+F_v^2}} \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix}$ is also tangent to surface, it is also unit, but in general it is not orthogonal to vector field $\mathbf{e}$, $\langle \mathbf{r}_u, \mathbf{r}_v \rangle = F_u F_v$. To find a second tangent vector field orthgonal to $\mathbf{e}$ we may consider the vector field $\mathbf{f}$ which is vector product $\mathbf{f} = \mathbf{n} \times \mathbf{e}$:

$$\mathbf{f} = \mathbf{n} \times \mathbf{e} = \frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \times \frac{1}{\sqrt{1+F_u^2}} \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} =$$

[13] In the previous section we calculated curvatures of cylinder, and sphere using derivation formulae. These calculations may be even easier to perform using just usual methods which we studied in the course of Geometry.

$$\frac{1}{\sqrt{(1+F_u^2+F_v^2)(1+F_u^2)}}\begin{pmatrix}-F_uF_v\\1+F_v^2\\F_v\end{pmatrix}\times$$

Since vector field $\mathbf{n}$ is orthogonal to surface, and it is unit, hence vector fields $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ form orthonormal basis, $\mathbf{e},\mathbf{f}$ are tangent. Thus we constructed orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ attached to the surface. We want to calculate curvatures at the origin a point $\mathbf{p}$ with coordinates $u=v=0$ Put the following condition: the surface $z=F(x,y)$ has extremum at the origin, i.e.

$$F_u\big|_{\mathbf{p}}=F_v\big|_{\mathbf{p}}=0\,,\quad (\mathbf{p}\text{ it has coordinates } u=v=0). \tag{6.69}$$

This conidition is not demanding. For every point A on the surface on the surface one can find adjusted Cartesian coordinates such that in these coordinates this surface will have extremum at the point A. On the other hand this condition drastically simplifies calculations. Note that if condition (6.69) is obeyed then at the point $\mathbf{p}$ vector fields $\mathbf{e},\mathbf{f},\mathbf{n}$ look in a very simple way:

$$\mathbf{e}\big|_{\mathbf{p}}=\mathbf{e}(u,v)_{u=v=0}=\begin{pmatrix}1\\0\\0\end{pmatrix},\mathbf{f}\big|_{\mathbf{p}}=\mathbf{f}(u,v)_{u=v=0}=\begin{pmatrix}0\\1\\0\end{pmatrix},\mathbf{n}\big|_{\mathbf{p}}=\mathbf{n}(u,v)_{u=v=0}=\begin{pmatrix}0\\0\\1\end{pmatrix},$$

Now obtain derivation formulae (at the point $\mathbf{p}$) We will calculate everything just at the point $\mathbf{p}$. Note that if condition (6.69) is obeyed then

$$d\mathbf{e}\big|_{\mathbf{p}}=d\left(\frac{1}{\sqrt{1+F_u^2}}\begin{pmatrix}1\\0\\F_u\end{pmatrix}_{\mathbf{p}}\right)=\begin{pmatrix}0\\0\\dF_u\end{pmatrix}_{u=v=0}=(\mathbf{n}dF_u)_{\mathbf{p}}\,,$$

$$d\mathbf{f}\big|_{\mathbf{p}}=d\left(\frac{1}{\sqrt{(1+F_u^2+F_v^2)(1+F_u^2)}}\begin{pmatrix}-F_uF_v\\1+F_v^2\\F_v\end{pmatrix}_{\mathbf{p}}\right)=\begin{pmatrix}0\\0\\dF_u\end{pmatrix}_{u=v=0}=(\mathbf{n}dF_v)_{\mathbf{p}}\,,$$

and

$$d\mathbf{n}\big|_{\mathbf{p}}=d\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\begin{pmatrix}-F_u\\-F_v\\1\end{pmatrix}_{\mathbf{p}}\right)=\begin{pmatrix}0\\0\\dF_u\end{pmatrix}_{u=v=0}=(-\mathbf{e}dF_u-\mathbf{f}dF_v)_{\mathbf{p}}\,,$$

since all other terms vanish at $u=v=0$. Comparing with derivation formula

$$d\begin{pmatrix}\mathbf{e}\\\mathbf{f}\\\mathbf{n}\end{pmatrix}=\begin{pmatrix}0&a&b\\-a&0&c\\-b&-c&0\end{pmatrix}d\begin{pmatrix}\mathbf{e}\\\mathbf{f}\\\mathbf{n}\end{pmatrix}$$

we see that forms a, b, c at origin are equal to

$$a\big|_{\mathbf{p}} = 0\,, b\big|_{\mathbf{p}} = dF_u\big|_{\mathbf{p}} = (F_{uu}du + F_{uv}dv)_{\mathbf{p}}\,, c\big|_{\mathbf{p}} = dF_v\big|_{\mathbf{p}} = (F_{vu}du + F_{vv}dv)_{\mathbf{p}}\,,$$

Now calculate the values of these forms on vectors $\mathbf{e}, \mathbf{f}$ at origin. We have that $\mathbf{b}(\mathbf{e}) = dF_u(\mathbf{e})_{\mathbf{p}} = (F_{uu}du + F_{uv}dv)(\mathbf{r}_u) = F_{uu}$. Analogously $\mathbf{b}(\mathbf{f}) = dF_u(\mathbf{f})_{\mathbf{p}} = (F_{uu}du + F_{uv}dv)(\mathbf{r}_v) = F_{uv}$, $c(\mathbf{e}) = dF_v(\mathbf{e})_{\mathbf{p}} = (F_{vu}du + F_{vv}dv)(\mathbf{r}_u) = F_{vu}$, and $c(\mathbf{f}) = dF_v(\mathbf{f})_{\mathbf{p}} = (F_{vu}du + F_{vv}dv)(\mathbf{r}_v) = F_{vv}$,

Hence we have that matrix of Weintegarten (shape) operator at the origin is equal to

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} F_{uu} & F_{uv} \\ F_{vu} & F_{vv} \end{pmatrix}_{\mathbf{p}}, K = \det S = F_{uu}F_{vv} - F_{uv}^2\,, \quad H = \operatorname{Tr} S = F_{uu} + F_{vv}$$

Theorem For surface $z = F(x, y)$, Wengarten (Shape) operator in extremum point is given by quadratic form (Hessian) of function.

Example

Consider surface defined by equation $z = Ax^2 + 2Bxy + y^2$, The point $x = y = 0$ is extremum point. All derivatives $F_u = 2Au + 2Bv, F_v = 2Bu + 2Cv$ vanish t origin.

Then Gaussian curvature at point $x = y = 0$ is equal to

$$K = F_{xx}F_{yy} - F_{xy}^2$$

Gaussian curvature at arbitrary point of surface $z = F(x, y)$ is equal to

$$K = \frac{F_{xx}F_{yy} - F_{xy}^2}{(1 + F_x^2 + F_y^2)^{3/2}}$$

6.4.1 Proof of the Theorem of parallel transport along closed curve

We are ready now to prove the Theorem. Recall that the Theorem states following:

If C is a closed curve on a surface M such that C is a boundary of a compact oriented domain $D \subset M$, then during the parallel transport of an arbitrary tangent vector along the closed curve C the vector rotates through the angle

$$\Delta\Phi = \angle\,(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = \int_D K d\sigma\,, \tag{6.70}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g}dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g}dudv$.

(see (**??**).

Recall that for derivation formula (6.29) we obtained structure equations

$$\begin{aligned} da + b \wedge c = 0 \\ db + c \wedge a = 0 \\ dc + a \wedge a = 0 \end{aligned} \tag{6.71}$$

We need to use only one of these equations, the equation

$$da + b \wedge c = 0\,. \tag{6.72}$$

This condition sometimes is called *Gauß condition.*

Let as always $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ be an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$ such that $\{\mathbf{e}, \mathbf{f}\}$ is an orthonormal basis in $T_{\mathbf{p}}M$ at every point of surface $\mathbf{p} \in M$. Then the Gauß condition (6.72) and equation (6.44) mean that for Gaussian curvature on the surface M can be expressed through the 2-form da and base vectors $\{\mathbf{e}, \mathbf{f}\}$:

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) \tag{6.73}$$

We use this formula to prove the Theorem.

Now calculate the parallel transport of an arbitrary tangent vector over the closed curve C on the surface M.

Let $\mathbf{r} = \mathbf{r}(u, v) = \mathbf{r}(u\alpha)$ $(\alpha = 1, 2,\ (u, v) = (u^1, v^1))$ be an equation of the surface M.

Let $u^\alpha = u^\alpha(t)$ $(\alpha = 1, 2)$ be the equation of the curve C. Let $\mathbf{X}(t)$ be the parallel transport of vector field along the closed curve C, i.e. $\mathbf{X}(t)$ is tangent to the surface M at the point $u(t)$ of the curve C and vector field $\mathbf{X}(t)$ is covariantly constant along the curve:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0$$

To write this equation in components we usually expanded the vector field in the coordinate basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and used Christoffel symbols of the connection $\Gamma^\alpha_{\beta\gamma}\colon \nabla_\beta \partial_\gamma = \Gamma^\alpha_{\beta\gamma}\partial_\alpha$.

Now we will do it in different way: *instead coordinate basis* $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ *we will use the basis* $\{\mathbf{e}, \mathbf{f}\}$. In the subsection 3.4.4 we obtained that the connection ∇ has the following appearance in this basis

$$\nabla_{\mathbf{v}}\mathbf{e} = a(\mathbf{v})\mathbf{f},\ ,\nabla_{\mathbf{v}}\mathbf{f} = -a(\mathbf{v})\mathbf{e} \tag{6.74}$$

(see the equations (6.37) and (6.38))

Let

$$\mathbf{X} = \mathbf{X}(u(t)) = X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t))$$

Lbe an expansion of tangent vector field $\mathbf{X}(t)$ over basis $\{\mathbf{e}, \mathbf{f}\}$. Let $\mathbf{v}$ be velocity vector of the curve C. Then the equation of parallel transport $\frac{\nabla \mathbf{X}(t)}{dt} = 0$ will have the following appearance:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0 = \nabla_{\mathbf{v}} \left(X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t)) \right) =$$

$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)\nabla_{\mathbf{v}}\mathbf{e}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) + X^2(t)\nabla_{\mathbf{v}}\mathbf{f}(u(t)) =$$

$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)a(\mathbf{v})\mathbf{f}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) - X^2(t)a(\mathbf{v})\mathbf{e}(u(t)) =$$

$$\left(\frac{dX^1(t)}{dt} - X^2(t)a(\mathbf{v})\right)\mathbf{e}(u(t)) + \left(\frac{dX^2(t)}{dt} + X^1(t)a(\mathbf{v})\right)\mathbf{f}(u(t)) = 0.$$

Thus we come to equation:

$$\begin{cases} \dot{X}^1(t) - a(\mathbf{v}(t))X^2 = 0 \\ \dot{X}^2(t) + a(\mathbf{v}(t))X^1 = 0 \end{cases}$$

There are many ways to solve this equation. It is very convenient to consider complex variable

$$Z(t) = X^1(t) + iX^2(t)$$

We see that

$$\dot{Z}(t) = \dot{X}^1(t) + i\dot{X}^2(t) = a(\mathbf{v}(t))X^2 - ia(\mathbf{v}(t)X^1 = -ia(\mathbf{v})Z(t),$$

i.e.

$$\frac{dZ(t)}{dt} = -ia(\mathbf{v}(t))Z(t) \tag{6.75}$$

The solution of this equation is:

$$Z(t) = Z(0)e^{-i\int_0^t a(\mathbf{v}(\tau))d\tau} \tag{6.76}$$

Calculate $\int_0^{t_1} a(\mathbf{v}(\tau))d\tau$ for closed curve $u(0) = u(t_1)$. Due to Stokes Theorem:

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da$$

Hence using Gauss condition (6.72) we see that

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da = -\int_D b \wedge c$$

Claim

$$\int_D b \wedge c = -\int_D da = \int K d\sigma\,. \qquad (6.77)$$

Theorem follows from this claim:

$$Z(t_1) = Z(0)e^{-i\int_C a} = Z(0)e^{i\int_D b\wedge C} \qquad (6.78)$$

Denote the integral $i\int_D b \wedge C$ by $\Delta\Phi$: $\Delta\Phi = i\int_D b \wedge C$. We have

$$Z(t_1) = X^1(t_1) + iX^2(t_1) = \left(X^1(0) + iX^2(0)\right) e^{i\Delta\Phi} = \qquad (6.79)$$

It remains to prove the claim. The induced volume form $d\sigma$ is 2-form. Its value on two orthogonal unit vector $\mathbf{e}, \mathbf{f}$ equals to 1:

$$d\sigma(\mathbf{e}, \mathbf{f}) = 1 \qquad (6.80)$$

(In coordinates u, v volume form $d\sigma = \sqrt{\det g} du \wedge dv$).

The value of the form $b \wedge c$ on vectors $\{\mathbf{e}, \mathbf{f}\}$ equals to Gaussian curvature according to (6.73). We see that

$$b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) = K d\sigma(\mathbf{e}, \mathbf{f})$$

Hence 2-forms $b \wedge c$, $-da$ and volume form $d\sigma$ coincide. Thus we prove (6.77).

6.5 Proof of the Theorem on curvature of surfaces given in conformal coordinates using derivation formulae

We return here to subsection 6.3 where we formulated the Theorem about Gaussian curvatureof surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ in conformal coordinates(6.64) . Let (u, v) be local conformal coordinates, and metric $G = \sigma(u, v)(du^2 + dv^2) = e^{\Phi}(du^2 + dv^2)$.

Consider vectors

$$\mathbf{e} = e^{-\frac{\Phi}{2}} \frac{\partial}{\partial u}\,, \quad \mathbf{f} = e^{-\frac{\Phi}{2}} \frac{\partial}{\partial v}\,, \quad \mathbf{n} = \mathbf{e} \times \mathbf{f}\,.$$

It is evident that $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ form orthonoromal basis:

$$\langle \mathbf{e}, \mathbf{e}\rangle = 1\,, \langle \mathbf{e}, \mathbf{f}\rangle = 0\,, \langle \mathbf{e}, \mathbf{n}\rangle = 0,\ \langle \mathbf{f}, \mathbf{f}\rangle = 1\, \langle \mathbf{f}, \mathbf{n}\rangle = 0\,, \langle \mathbf{n}, \mathbf{n}\rangle = 1\,.$$

Consider derivation formula (6.29) for this basis:

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \qquad (6.81)$$

To calculate Gaussian curvature we need to calculate 1-form a in this equations since accordind equations (6.44) and (6.34) $K = b \wedge c(\mathbf{e}, \mathbf{f})$ and $da + b \wedge c = 0$, i.e. $K = -da(\mathbf{e}, \mathbf{f})$. Now calculate 1-form a. We have

$$d\mathbf{e} = d\left(e^{-\frac{\Phi}{2}}\mathbf{r}_u\right) = a\mathbf{f} + b\mathbf{n}\,.$$

Taking scalar product of this equation of $\mathbf{f}$ we come to

$$a = \langle d\mathbf{e}, \mathbf{f}\rangle = \left\langle d\left(e^{-\frac{\Phi}{2}}\mathbf{r}_u\right), e^{-\frac{\Phi}{2}}\mathbf{r}_v\right\rangle. \tag{6.82}$$

Calculate it. Since $\langle \mathbf{r}_u, \mathbf{r}_v\rangle = 0$ then

$$\left\langle d\left(e^{-\frac{\Phi}{2}}\mathbf{r}_u\right), e^{-\frac{\Phi}{2}}\mathbf{r}_v\right\rangle = e^{-\Phi}\langle d\mathbf{r}_u, \mathbf{r}_v\rangle = e^{-\Phi}\langle \mathbf{r}_{uu}, \mathbf{r}_v\rangle du + e^{-\Phi}\langle \mathbf{r}_{uv}, \mathbf{r}_v\rangle dv\,.$$

Now using the fact that $\langle \mathbf{r}_v, \mathbf{r}_v\rangle = \langle \mathbf{r}_u, \mathbf{r}_u\rangle = e^{\Phi}$ and $\langle \mathbf{r}_u, \mathbf{r}_v\rangle = 0$ calculate $\langle \mathbf{r}_{uv}, \mathbf{r}_v\rangle$ and $\langle \mathbf{r}_{uu}, \mathbf{r}_v\rangle$. We have

$$\langle \mathbf{r}_{uv}, \mathbf{r}_v\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle \mathbf{r}_v, \mathbf{r}_v\rangle = \frac{1}{2}\frac{\partial}{\partial u}\left(e^{\Phi}\right) = \frac{1}{2}\Phi_u e^{\Phi}$$

and

$$\langle \mathbf{r}_{uu}, \mathbf{r}_v\rangle = \frac{\partial}{\partial u}\langle \mathbf{r}_u, \mathbf{r}_v\rangle - \langle \mathbf{r}_u, \mathbf{r}_{uv}\rangle = 0 - \langle \mathbf{r}_u, \mathbf{r}_{uv}\rangle = -\frac{1}{2}\frac{\partial}{\partial v}\langle \mathbf{r}_v, \mathbf{r}_v\rangle = -\frac{1}{2}\frac{\partial}{\partial v}\left(e^{\Phi}\right) = -\frac{1}{2}\Phi_v e^{\Phi}$$

Hence we see that 1-form a in (6.82) is equal to

$$a = \langle d\mathbf{e}, \mathbf{f}\rangle = \left\langle d\left(e^{-\frac{\Phi}{2}}\mathbf{r}_u\right), \left(e^{-\frac{\Phi}{2}}\mathbf{r}_v\right)\right\rangle = e^{-\Phi}\langle \mathbf{r}_{uu}, \mathbf{r}_v\rangle du + e^{-\Phi}\langle \mathbf{r}_{uv}, \mathbf{r}_v\rangle dv = \frac{1}{2}(\Phi_u dv - \Phi_v du)\,, \tag{6.83}$$

and 2-form

$$da = d\left(\frac{1}{2}(\Phi_u dv - \Phi_v du)\right) = \frac{1}{2}(\Phi_{uu} du \wedge dv - \Phi_{vv} dv \wedge du) = \frac{1}{2}\left(\Phi_{uu} + \Phi_{vv}\right) dv \wedge du\,.$$

Now using Gauss formula (6.34) and (6.44) we come to

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) = -\frac{1}{2}\left(\Phi_{uu} + \Phi_{vv}\right) du \wedge dv\left(e^{-\frac{\Phi}{2}}\frac{\partial}{\partial u}, e^{-\frac{\Phi}{2}}\frac{\partial}{\partial v}\right) =$$

$$-\frac{e^{-\frac{\Phi}{2}}}{2}\left(\Phi_{uu} + \Phi_{vv}\right) du \wedge dv\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = -\frac{e^{-\Phi}}{2}\left(\Phi_{uu} + \Phi_{vv}\right) = -\frac{e^{-\Phi}}{2}\Delta\Phi,$$

where Laplacian $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

It is useful to write down this fiormula in complex coordinates Write down the formula in holomorphic coordinates: $z = u + iv, \bar{z} = u - iv$. We have that

$$K = -\frac{e^{-\Phi}}{2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\Phi = -\frac{e^{-\Phi}}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\Phi = -2e^{-\Phi}\frac{\partial^2\Phi}{\partial\bar{z}\partial z}\,, \tag{6.84}$$

(for definition of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ see (6.67)). This expression is sometimes very convenient for calculations.

Example Consider sphere of radius 1 in stereographic coordinates. Then $G = \frac{4(du^2+dv^2 0}{(1+u^2+v^2)^2)}$. In complex coordinates $G = \frac{4dzd\bar{z}}{(1+z\bar{z})^2} = e^\Phi dzd\bar{z}$ with $e^\Phi = \frac{4}{(1+z\bar{z})^2}$, i.e. $\Phi = \log 4 - 2\log(1+z\bar{z})$. We see that $\Phi_z = -\frac{2}{1+z\bar{z}}$ and $\Phi_{z\bar{z}} = -\frac{2}{(1+z\bar{z})^2}$, i.e. $K = -2e^{-\Phi}\Phi_{\bar{z}z} = 1$.

Exercise Let $z = f(w)$ be an holomorphic changing of complex coordinates. Due to Theorem new coordinates u', v' ($w = u' + iv'$, $z = u + iv$) are isothermal coordinates too: If

$$G = e^\Phi(du^2 + dv^2) = e^\Phi dzd\bar{z} = e^{\Phi'}dwd\bar{w} = e^{\Phi'}(du'^2 + dv'^2)\,.$$

It is very illuminating to check straightforwardly that calculating of Gaussian curvature in new coordinates we will come to the same answer. Do it. According to (6.68) we see that $e^\Phi dzd\bar{z} = e^\Phi f_w\overline{f_w}dwd\bar{w}$, i.e. $\Phi' = \Phi + \log f_w + \log\overline{f_w}$. Hence

$$\frac{\partial^2\Phi'}{\partial\bar{w}\partial w} = \frac{\partial^2\Phi}{\partial\bar{w}\partial w} + \frac{\partial^2}{\partial\bar{w}\partial w}\left(\log f_w + \log\overline{f_w}\right)\,.$$

Notice that the function $\log f_w$ is holomorphic function $\Leftrightarrow \frac{\partial}{\partial\bar{w}}\log f_w = 0$ and the function $\log\overline{f_w}$ is anti-holomorphic function $\Leftrightarrow \frac{\partial}{\partial w}\log\overline{f_w} = 0$ too. Hence

$$\frac{\partial^2}{\partial\bar{w}\partial w}\left(\log f_w + \log\overline{f_w}\right) = 0\,.$$

This implies that

$$\frac{\partial^2\Phi'}{\partial\bar{w}\partial w} = \frac{\partial^2\Phi}{\partial\bar{w}\partial w}\,.$$

Again using the fact that functions $z = f(w)$ and $z_w = f_w$ are holomorphic functions we see that

$$\frac{\partial^2\Phi'}{\partial\bar{w}\partial w} = \frac{\partial^2\Phi}{\partial\bar{w}\partial w} = \frac{\partial}{\partial\bar{w}}\left(\Phi_z f_w\right) = \frac{\partial\Phi_z}{\partial\bar{w}}f_w = \Phi_{\bar{z}z}f_w\overline{f_w}.$$

Finally we come to

$$K = -2e^{-\Phi'}\frac{\partial^2\Phi'}{\partial\bar{w}\partial w} = -2e^{-\Phi-\log f_w-\log\overline{f_w}}\Phi_{\bar{z}z}f_w\overline{f_w} = -2e^{-\Phi}\frac{\partial^2\Phi}{\partial\bar{z}\partial z}\,.$$

Thus we check by straightforward calculations that Gaussian curvature remains the same.In these calculations In these calculations we used intensively properties (6.67) of holomorphic and anti-holomorphic functions.

6.6 Straightforward proof of the Proposition: $K = \frac{R}{2} = R_{1212} \det g$ (??)

(Content of this paragraph is similar to the content of the solution of exercise 6 in Homework 6. It is useful to read them both.)

We prove this fact by direct calculations. The plan of calculations is following:

Let M be a surface in $\mathbf{E}^3$ For an arbitrary point $\mathbf{p}$ of the surface M we consider Cartesian coordinates x, y, z such that origin coincides with the point $\mathbf{p}$ and coordinate plane OXY is the palne attached at the surface at the point $\mathbf{p}$ and the axis OZ is orthogonal to the surface. In these coordinates calculations become easy. (See the subsection 6.4) The surface M in these Cartesian coordinates can be expressed by the equation

$$\begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \tag{6.85}$$

where $F(u, v)$ has local extremum at the point $u = v = 0$. We calculated in subsection 6.4 Gaussian curvature at this point: Gaussian curvature at the point $\mathbf{p}$ equals to

$$K = \det S = F_{uu}F_{vv} - F_{uv}^2 \,. \tag{6.86}$$

(all the derivatives at the origin).

Now it is time to calculate the Riemann curvature tensor at the origin.

First of all recall the expression for Riemannian metric for the surface M in a vicinity of origin is

$$G = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle & \langle \mathbf{r}_u, \mathbf{r}_v \rangle \\ \langle \mathbf{r}_v, \mathbf{r}_u \rangle & \langle \mathbf{r}_v, \mathbf{r}_v \rangle \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_v F_u & 1 + F_v^2 \end{pmatrix} \,. \tag{6.87}$$

This immediately follows from the expression for basic vectors $\mathbf{r}_u, \mathbf{r}_v$

Note that Riemannian metric g_{ik} at the point $u = v = 0$ is defined by unity matrix $g_{uu} = g_{vv} = 1$, $g_{uv} = g_{vu} = 0$ since $\mathbf{p}$ is extremum point: $G = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_v F_u & 1 + F_v^2 \end{pmatrix} \big|_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since $\mathbf{p}$ is stationary point, extremum ($F_u = F_v = 0$). Hence the components of the tensor $R^i{}_{kmn}$ and $R_{ikmn} = g_{ij} R^j{}_{kmn}$ at the point $\mathbf{p}$ are the same.

For 2-dimensional surface Riemann curvature tensor has oessentially only one not-vanishing component R^1_{212}. All other components vanish all are equal up to a sign to this component:

$$R_{1212} = -R_{2112} = R_{2121}\, R_{1112} = \cdots = R_{2111} = 0 \,.$$

So in fact we need to calculate only one component, the component R_{1212}.

In our calculations we will use the fact that Riemannian metric at the point **p** is defined by unity matrix, and that first derivatives of metric at **p** vanish, i.e.Christoffel symbols in coordinates u, v vanish at the point **p**.

Recall that the components of $R^i{}_{kmn}$ are defined by the formula

$$R^i{}_{kmn} = \partial_m \Gamma^i_{nk} + \Gamma^i_{mp}\Gamma^p_{nk} - \partial_n \Gamma^i_{mk} - \Gamma^i_{np}\Gamma^p_{mk} \,.$$

(see equation (**??**)). Notice that at the point **p** not only the matrix of the metric g_{ik} equals to unity matrix, but more: Christoffel symbols vanish at this point in coordinates u, v since the derivatives of metric at this point vanish. (Why they vanish: this immediately follows from Levi-Civita formula applied to the metric (6.87), see also in detail the file "The solution of the problem in the coursework"). Hence to calculate R^i_{kmn} at the point **p** one can consider much more simple formula than formula (**??**):

$$R^i{}_{kmn}|_{\mathbf{p}} = \partial_m \Gamma^i_{nk}|_{\mathbf{p}} - \partial_n \Gamma^i_{mk}|_{\mathbf{p}}$$

Try to continue calculations in a more "economical" way. Due to Levi-Civita formula

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right)$$

Since metric g_{ik} equals to unity matrix $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ at the point **p** hence g^{ij} is unity matrix also:

$$g^{ik}|_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^{ik} \,.$$

(We denote δ^{ik} the unity matrix: all diagonal components equal to 1, all other components equal to zero. (It is so called Kronecker symbols)) Moreover we know also that all the first derivatives of the metric vanish at the point **p**:

$$\frac{\partial g_{ik}}{\partial x^m}|_{\mathbf{p}} = 0 \,.$$

Hence it follows from the formulae above that for an arbitrary indices i, j, k, m, n

$$\frac{\partial}{\partial x^i} \left(g^{km} \frac{\partial g_{pr}}{\partial x^j} \right) \Big|_{\mathbf{p}} = \frac{\partial g^{km}}{\partial x^i}\Big|_{\mathbf{p}} \frac{\partial g_{pr}}{\partial x^j}\Big|_{\mathbf{p}} + g^{km}\Big|_{\mathbf{p}} \frac{\partial^2 g_{pr}}{\partial x^i \partial x^j}\Big|_{\mathbf{p}} = \delta^{km} \frac{\partial^2 g_{pr}}{\partial x^i \partial x^j}\Big|_{\mathbf{p}} \,.$$

Now using the Levi-Civita formula for the Christoffel symbols of connection:

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right)$$

we come to $\partial_n \Gamma^i_{mk}|_{\mathbf{p}} = \frac{\partial}{\partial x^n}\left(\frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)\right)|_{\mathbf{p}} =$

$$\frac{1}{2}\delta^{ij}\left(\frac{\partial^2 g_{jm}}{\partial x^n \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^n \partial x^m} - \frac{\partial^2 g_{mk}}{\partial x^n \partial x^j}\right)|_{\mathbf{p}} . \tag{6.88}$$

since first derivatives of metric vanish at the point $\mathbf{p}$.

Now using this formula we are ready to calculate Riemann curvature tensor $R^i{}_{kmn}$. Remember that it is enough to calculate $R^1{}_{212}$ and $R^1{}_{212} = R_{1212}$ at the point $\mathbf{p}$ since $g_{ik} = \delta_{ik}$ at the point $\mathbf{p}$. We have that at $\mathbf{p}$ $R^1{}_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}}$. Now using equation (6.88) we come to

$$R^1_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}} = \frac{1}{2}\frac{\partial}{\partial x^1}\left(2\frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) - \frac{1}{2}\frac{\partial}{\partial x^2}\left(\frac{\partial g_{11}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^1}\right)\mathbf{p} =$$

$$\frac{\partial^2 g_{uv}}{\partial u \partial v}|_{\mathbf{p}} - \frac{1}{2}\frac{\partial^2 g_{vv}}{\partial u^2}|_{\mathbf{p}} - \frac{1}{2}\frac{\partial^2 g_{uu}}{\partial v^2}|_{\mathbf{p}} \tag{6.89}$$

Now return to the surface (6.85) We have that $g_{uu} = 1 + F_u^2, g_{uv} = F_u F_v$ and $g_{vv} = 1 + F_v^2$,hence

$$\frac{\partial^2 g_{uv}}{\partial u \partial v}|_{\mathbf{p}} = (F_{uu}F_v + F_u F_{uv})_v = F_{uu}F_{vv} + F_{uv}^2 ,$$

$$\frac{\partial^2 g_{vv}}{\partial u^2}|_{\mathbf{p}} = (2F_v F_{vu})_u = 2F_{uv}^2 ,$$

$$\frac{\partial^2 g_{uu}}{\partial v^2}|_{\mathbf{p}} = (F_{uu}F_u)_v = 2F_{uv}^2 ,$$

Hence

$$R_{1212}|_{\mathbf{p}} = \left((F_{uu}F_{vv} + F_{uv}^2) - 2F_{uv}^2\right)_{\mathbf{p}} = \left(F_{uu}F_{vv} - F_{uv}^2\right)_{\mathbf{p}} = K_{\mathbf{p}} .$$

The proof is finished: we showed by straightforward calculations that R^1_{212} is equal to Gaussian curvature K. On the other hand at the point $\mathbf{p}$, $\det g = 1$. Thus we come to the statement of Proposition**??**.

Repeat again: all other components of Riemann curvature tensor R^i_{kmn} are equal to R_{1212} up to a sign or vanish. Hence we calculated Riemann curvature tensor at the point $\mathbf{p}$ and showed that it is essentially is defined by Gaussian curvature.

It is important to note that in our calculations of R_{1212} (see formula (6.89)) we used only the fact that Riemannian metric at the point $\mathbf{p}$ is defined by unity matrix, and all first derivatives at this point vanish.(see also Statement 1 in the solution of exercise 6 of Homework 9)

6.6.1 Curvature of surfaces in $\mathbf{E}^3$. Theorema Egregium again (another proof)

For surfaces in $\mathbf{E}^3$ Gaussian curvature is equal to half of scalar curvature:

$$K = \frac{R}{2}\,, \tag{6.90}$$

where $R = R^i_{kin} g^{kn}$ is scalar curvature of Riemann curvature tensor.

Equation (6.93) is the fundamental relation which claims that the Gaussian curvature (the magnitude defined in terms of External observer) equals to the scalar curvature (up to a coefdficient), the magnitude defined in terms of Internal Observer. This gives us another proof of Theorema Egregium.

Prove this formula.

Express Riemannian curvature of surfaces in $\mathbf{E}^3$ in terms of derivation formula (6.29).

Consider derivation formula (6.29) for the orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n},\}$ adjusted to the surface M:

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}\,, \tag{6.91}$$

where as usual $\mathbf{e}, \mathbf{f}, \mathbf{n}$ vector fields of unit length which are orthogonal to each other and $\mathbf{n}$ is orthogonal to the surface M. As we know the induced connection on the surface M is defined by the formula (6.37) and (6.38):

$$\nabla_{\mathbf{Y}}\mathbf{e} = (d\mathbf{e}(\mathbf{Y}))_{\text{tangent}} = a(\mathbf{X})\mathbf{f}\,, \nabla_{\mathbf{Y}}\mathbf{f} = (d\mathbf{f}(\mathbf{Y}))_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}\,, \tag{6.92}$$

According to the definition of curvature calculate

$$R(\mathbf{e}, \mathbf{f})\mathbf{e} = \nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e}\,.$$

Using these formulae one can calculate straightforwardly that for surfaces in $\mathbf{E}^3$ Gaussian curvature is equal to half of the scalar curvature:

$$K = \frac{R}{2} \tag{6.93}$$

Detailed calculations are following:

Note that since the induced connection is symmetrical connection then:

$$\nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} - [\mathbf{e}, \mathbf{f}] = 0\,.$$

hence due to (6.92)

$$[\mathbf{e}, \mathbf{f}] = \nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} = -a(\mathbf{e})\mathbf{e} - a(\mathbf{f})\mathbf{f}$$

Thus we see that $R(\mathbf{e}, \mathbf{f})\mathbf{e} =$

$$\nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e} = \nabla_{\mathbf{e}}\left(a(\mathbf{f})\mathbf{f}\right) - \nabla_{\mathbf{f}}\left(a(\mathbf{e})\mathbf{f}\right) + \nabla_{a(\mathbf{e})\mathbf{e}+a(\mathbf{f})\mathbf{f}}\mathbf{e} =$$

$$\partial_{\mathbf{e}} a(\mathbf{f})\mathbf{f} + a(\mathbf{f})\nabla_{\mathbf{e}}\mathbf{f} - \partial_{\mathbf{f}} a(\mathbf{e})\mathbf{f} - a(\mathbf{e})\nabla_{\mathbf{f}}\mathbf{f} + a(\mathbf{e})\nabla_{\mathbf{e}}\mathbf{e} + a(\mathbf{f})\nabla_{\mathbf{f}}\mathbf{e} =$$

$$\partial_{\mathbf{e}} a(\mathbf{f})\mathbf{f} - a(\mathbf{f})a(\mathbf{e})\mathbf{e} - \partial_{\mathbf{f}} a(\mathbf{e})\mathbf{f} + a(\mathbf{e})a(\mathbf{f})\mathbf{e} + a(\mathbf{e})a(\mathbf{e})\mathbf{f} + a(\mathbf{f})a(\mathbf{f})\mathbf{f} =$$

$$[\partial_{\mathbf{e}} a(\mathbf{f})\mathbf{f} - \partial_{\mathbf{f}} a(\mathbf{e})\mathbf{f} - a\,[-a(\mathbf{e})\mathbf{e} - a(\mathbf{f})\mathbf{f}]]\,\mathbf{f} =$$

$$= [\partial_{\mathbf{e}} a(\mathbf{f})\mathbf{f} - \partial_{\mathbf{f}} a(\mathbf{e})\mathbf{f} - a\,([\mathbf{e},\mathbf{f}])]\,\mathbf{f} = da(\mathbf{e},\mathbf{f})\mathbf{f}\,.$$

Recall that we established in 6.73 that for Gaussian curvature K

$$K = b \wedge c(\mathbf{e},\mathbf{f}) = -da(\mathbf{e},\mathbf{f})$$

Hence we come to the relation:

$$R(\mathbf{e},\mathbf{f})\mathbf{e} = da(\mathbf{e},\mathbf{f}) = -K\mathbf{f}\,.$$

This means that

$$R^2_{112} = -K$$

(in the basis $\mathbf{e},\mathbf{f}$), i.e. the scalar curvature

$$R = 2R_{1212} = 2K$$

Thus we come to equation (6.93).

The proof of Theorema Egregium by straightforward calculations see in the previous subsection.