Riemannian Geometry

2020

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 26 March 4pm

Write solutions in the provided spaces.

STUDENT'S NAME:

1

Consider a surface M, the upper sheet of the cone

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = 2h \end{cases}, \qquad 0 \leq \varphi < 2\pi\,,\ h > 0\,. \tag{1}$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in $\mathbf{E}^3$ in coordinates (h,φ).

Show that this surface is locally Euclidean by giving an example of local coordinates (u,v), which are Euclidean coordinates.

Find the length of the shortest curve which belongs to the surface M, starts at the point $(h_0, 0, 2h_0)$ and ends at the point $(-h_0, 0, 2h_0)$.

[3 marks]

2

Consider a sphere S^2 of the radius a in spherical coordinates

$$\mathbf{r}(\theta,\varphi)\colon \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\ 0 < theta < \pi\,. \tag{1}$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in $\mathbf{E}^3$ in spherical coordinates (θ,φ).

Give an example of non-identical transformation which is the isometry of the sphere.

Consider two points $A = (a\sin\theta_0, 0, a\cos\theta_0)$ $B = (-a\sin\theta_0, 0, a\cos\theta_0)$ on this sphere.

Calculate the length of the arc of the latitude $\begin{cases} \varphi = t \\ \theta = \theta_0 \end{cases}$, $0 \le t \le \pi$ which connects this points.

Explain why this is not the shortest curve on the sphere which connects points A and B.

Give an argument explaining why sphere is not locally Euclidean.

[3 marks]

3

Recall that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2} .$$

(a) Give an example of non-identical isometry, i.e. a non-identitical transformation of coordinates u, v which preserves this metric.

(b) Give an example of a non-linear transformation of coordinates u, v which preserves this metric.
(*Hint: You may find this transformation considering transformations of the sphere.*)

(**c**) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to this metric.
Explain why the length of this curve does not depend on a.

[4 marks]

4

Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

[2 marks]

5

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-a^2x^2 - a^2y^2}\left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

[1 marks]

6

Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

Consider in the Lobachevsky plane the domain D defined by

$$D = \{x, y\colon \quad x^2 + y^2 \geq 1\,,\ -a \leq x \leq a\}\,,$$

where a is a parameter ($0 < a < 1$).

Find the area of the domain D (with respect to the metric G).

Consider the points $A_t = (-a, t)$, $B_t = (a, t)$ on the vertical lines delimiting the domain D. Show that the distance between these points tends to 0 if $t \to \infty$.

[3 marks]

7

Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v), $\nabla_{\frac{\partial}{\partial u}}\left(u^2 \frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

[1 marks]

8

a) Let ∇ be an arbitrary connection on n-dimensional manifold M and let $\{\Gamma^i_{km}(x)\}$ be the Christoffel symbols of this connection. Let let $\omega = \omega_i(x)dx^i$ be a differential form on M. Show that

$$\tilde{\Gamma}^i_{km} = \Gamma^i_{km} + \delta^i_k \omega_m$$

are Christoffel symbols of the new connection.

(b) Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$.

Show that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some new connection.

Explain why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ are not Christoffel symbols for some connection.

[3 marks]