Differentiable Manifolds

it is a draft of lecture notes of H.M. Khudaverdian [1].
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# Contents



[1] it is based on the lecture course of T.Voronov

# 1 Differentiable manifolds and smooth maps

Roughly, "manifolds" are sets where one can introduce coordinates. Before giving precise definitions, let us discuss first the fundamental idea of coordinates. What are coordinates?

## 1.1 Coordinates on familiar spaces. Examples.

**Example 1.1.** Standard coordinates on $\mathbb{R}^n$: $\mathbb{R}^n \ni \mathbf{x} \leftrightarrow (x^1, \ldots, x^n)$. In particular, the standard coordinates $x, y, z$ on $\mathbb{R}^3$ (traditional notation).

**Example 1.2.** A linear change of coordinates on $\mathbb{R}^n$. E.g. for $\mathbb{R}^2$ "new" coordinates $x', y'$ are such that $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-degenerate matrix.

**Example 1.3.** Polar coordinates on $\mathbb{R}^2$ and spherical coordinates on $\mathbb{R}^3$.

For example, $\mathbb{R}^2 \ni \mathbf{x} \leftrightarrow (r, \varphi)$, where $x = r\cos\varphi, y = r\sin\varphi$ ($0 < r < \infty$ and $0 < \varphi < 2\pi$ or $-\pi < \varphi < \pi$). Note: coordinates $(r, \varphi)$ 'serve' not for the whole $\mathbb{R}^2$, but only for a part of it (an open subset).

For spherical coordinates $\mathbb{R}^3 \ni \mathbf{x} \leftrightarrow (r, \theta, \varphi)$, $0 < \theta < \pi$, $0 < \varphi < 2\pi$, and $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi,\ z = \mathbf{r}\cos\theta$. The same note: coordinates $(r, \theta, \varphi)$ 'serve' not for the whole $\mathbb{R}^3$.

**Example 1.4.** (stereographic coordinate) Consider the circle $S^1\colon x^2+y^2=1$ in $\mathbb{R}^2$. Consider the straight line which passes through the point $N=(0,1)$ (north pole of $S^1$) and he point $(x,y)$ on the circle. It intersects $x$-axis at the point $(u,0)$ where where $u=\frac{x}{1-y}$. It is so called stereographic coordinate on the circle. One can see that $x=\frac{2u}{u^2+1}, y=\frac{u^2-1}{u^2+1}$. This coordinate is "good" for all points of the circle except the "north pole". One can consider another stereographic coordinate $u'\in\mathbb{R}$, where $u'=\frac{x}{1+y}$, and conversely $x=\frac{2u'}{u'^2+1}, y=\frac{1-u'^2}{u'^2+1}$. This coordinate is good for all points of the circle except the point $S=(0,1)$ (the "south pole"). (See for details Homework)

**Example 1.5.** Similarly, stereographic coordinates can be defined for the unit sphere $S^2$ in $\mathbb{R}^3$ and, more generally, for $S^n\subset\mathbb{R}^{n+1}$ (See Homework).

**Example 1.6.** Another way of introducing a coordinate on $S^1$ is to consider the polar angle $\varphi$. It is defined initially up to an integral multiple of $2\pi$. To make it single-valued, we may restrict $0<\varphi<2\pi$ and thus we have to exclude the point $(1,0)$. We may introduce $\varphi'$ so that $-\pi<\varphi'<\pi$ and thus we have to exclude the point $(-1,0)$.

**Example 1.7.** Similarly, to obtain coordinates on $S^2\subset\mathbb{R}^3$, one may use the angles $\theta,\varphi$ making part of the spherical coordinates on $\mathbb{R}^3$. (To be able to define such angular coordinates as single-valued functions, certain points have to be excluded from the sphere. To cover the whole $S^2$, it will be necessary to consider several angular coordinate systems, each defined in a particular domain.)

*To deal with the next example you may consider $n=2,3$ or even $n=1$.*

**Example 1.8.** Recall the notion of a projective space. The *real projective space* $\mathbb{R}P^n$ is defined as the set of all straight lines through the origin in $\mathbb{R}^{n+1}$. Fix a hyperplane (a plane of dimension $n$) $H\subset\mathbb{R}^{n+1}$ **not** through the origin. For example, it is possible to take the hyperplane $x^{n+1}=1$. Each line through the origin $O$ intersects $H$ at a unique point, except for the lines parallel to $H$, which do not intersect $H$. The hyperplane $H$ can be identified with $\mathbb{R}^n$ by dropping the last coordinate $x^{n+1}=1$. Therefore the projective space $\mathbb{R}P^n$ can be visualised as the ordinary $n$-dimensional space $\mathbb{R}^n$ 'completed' by adding extra points to it. Notice that these extra points correspond to the straight lines through the origin in $\mathbb{R}^n\subset\mathbb{R}^{n+1}$ (considered as the coordinate hyperplane $x^{n+1} = 0$). Hence they make $\mathbb{R}P^{n-1}$, and we have

$$\mathbb{R}P^n = \mathbb{R}^n \cup \mathbb{R}P^{n-1} = \mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \ldots \cup \mathbb{R}^1 \cup \mathbb{R}^0$$

(where $\mathbb{R}^0$ is a single point). This construction introduces a coordinate system on the part $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$ of $\mathbb{R}P^n$. An inclusion $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$ is equivalent to a choice of hyperplane $H$ in $\mathbb{R}^{n-1}$. To cover by coordinates a different part of $\mathbb{R}P^n$, one has to choose a different $H$. It is not difficult to see that by considering the $n+1$ coordinate hyperplanes $x^k = 1$ as $H$, where $k = 1, \ldots, n+1$, we obtain $n+1$ coordinate systems covering together the whole $\mathbb{R}P^n$.

**Example 1.9.** The *complex projective space* $\mathbb{C}P^n$ is defined similarly to $\mathbb{R}P^n$ (with real numbers replaced by complex numbers). One can introduce coordinates into $\mathbb{C}P^n$ in the same way as above.

## 1.2 Definition of a manifold.

Recall that a set $V \subset \mathbb{R}^n$ is *open* if for each point $\mathbf{x} \in V$ there is an open $\varepsilon$-neighborhood entirely contained in $V$. (In greater detail, there $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset V$, where $B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{y}| < \varepsilon\}$. In other words, $B_\varepsilon(\mathbf{x})$ is an open ball of radius $\varepsilon$ with center at $\mathbf{x}$.)

There are many reasons why open sets in $\mathbb{R}^n$ are important. For us the main motivation is differential calculus, where one studies how the function changes if its argument is given a small increment, i.e., a given initial value of the argument is replaced by adding a small vector (which can point in an arbitrary direction). Therefore its is necessary to be able to consider a function on a whole neighborhood of any given point. So domains of definitions of functions have to be open if we wish to apply to them differential calculus.

Let $X$ be an abstract set. Fix a natural number $n$.

Let $U$ be a subset on $X$.

A *chart* $(U, \varphi)$ on $X$ is a bijective map $\varphi\colon U \to V$, where $V \subset \mathbb{R}^n$ is an open set in $\mathbb{R}^n$. (Recall that a map between two sets is bijective means that it establishes one-one correspondence between these sets.) The inverse map $\varphi^{-1}\colon V \to X$ is an injection of open domain $V$ in $X$. There is a one-to-one correspondence between points in $U \subset X$ and arrays $(x^1, \ldots, x^n) \in V \subset \mathbb{R}^n$ given by the maps $\varphi$ and $\varphi^{-1}$:

$$X \supset U \ni \mathbf{x},\ \varphi(\mathbf{x}) = (x^1, \ldots, x^n) \in V \subset \mathbb{R}^n\,.$$

The numbers $\varphi(\mathbf{x}) = (x^1, \ldots, x^n)$ are *coordinates* of a point $\mathbf{x} \in U \subset X$. A chart $\varphi$ on $U \subset X$ is a *coordinate system* on $X$. It is also called a 'local' coordinate system to emphasize that $\varphi$ is defined only for a subset $U \subset X$.

An *atlas* $\mathcal{A}$ on $X$ is a collection of charts, $\mathcal{A} = (U_\alpha, \varphi_\alpha)$, where $\varphi_\alpha \colon U_\alpha \to V_\alpha \subset \mathbb{R}^n$ for all $\alpha$, such that the subspaces $\{U_\alpha\}$ cover the whole space $X$:

$$X = \bigcup_\alpha U_\alpha\,.$$

It is convenient to think that $\mathbb{R}^n$'s for different $V_\alpha$ are 'different copies' of the space $\mathbb{R}^n$ and denote them, respectively, by $\mathbb{R}^n_{(\alpha)}$, so that we have $V_\alpha \subset \mathbb{R}^n_{(\alpha)}$. One should keep in mind a geographical atlas, pages of which correspond to different $\mathbb{R}^n_{(\alpha)}$ (geographical 'maps' of the Earth corresponding to mathematical 'charts').

Consider sets $U_\alpha$ and $U_\beta$ such that $U_\alpha \cap U_\beta \neq \varnothing$. Any point $\mathbf{x} \in U_\alpha \cap U_\beta$ has two coordinate descriptions: $\varphi_\alpha(\mathbf{x}) = (x^1_{(\alpha)}, \ldots, x^n_{(\alpha)})$ and $\varphi_\beta(\mathbf{x}) = (x^1_{(\beta)}, \ldots, x^n_{(\beta)})$. There is a *transition* map

$$\Psi_{\alpha\beta}(x^1, \ldots, x^n) = \varphi_\alpha \circ \varphi_\beta^{-1}(x^1, \ldots, x^n) \colon \ \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \quad (1.1)$$

$$\varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta), \qquad (x^1_\beta, \ldots, x^n_\beta) \mapsto (x^1_\alpha, \ldots, x^n_\alpha),$$

which we call the *change of coordinates* between charts $\varphi_a$ and $\varphi_\beta$, or transition functions form coordinates $((x^1_\beta, \ldots, x^n_\beta)$ to coordinates $(x^1_\alpha, \ldots, x^n_\alpha)$. The family of transition functions $\{\Psi_{\alpha\beta}\}$ are defined on domains of $\mathbb{R}^n$ and take values in domains of $\mathbb{R}^n$. Thus we can apply technique of calculus

**Definition 1.1.** An atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$, $(\varphi_\alpha \colon\ U_\alpha \to X)$ is *differentiable* or *smooth* if all sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ are open and the functions $\{\Psi_{\alpha\beta}\}$ of changing of coordinates $(\Psi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ are differentiable (smooth) functions.

**Remark 1.1.** "Smoothness" in differential geometry usually means having as many derivatives as necessary, unless we are specifically interested in finding the minimal order of differentiability for a particular problem. Therefore, in these lectures by *smooth* or *differentiable* (which we use interchangeably) will always mean the class $C^\infty$, i.e., infinitely many continuous derivatives.

**Definition 1.2.** A *differentiable manifold*, or *smooth manifold* (shortly: *manifold*) is a set $X$ endowed with a smooth atlas. The number $n$ is called the *dimension.*

Besides letters such as $X$, $Y$ and $Z$, other traditional letters for denoting manifolds are $M$, $N$, $P$, and $Q$.

The dimension of a manifold $M$ is often indicated by a superscript, e.g., $M = M^n$.

Strictly speaking, a manifold is not just a set, but a pair consisting of a set, say, $M$ and a smooth atlas on $M$, as described above. With a common abuse of language, we shall speak of a 'manifold $M$', with a certain atlas implicitly understood.

In view of the remark above it is clear that one can define manifolds of a particular class of smoothness $C^k$, for a fixed given $k$ (i.e., $k$ continuous derivatives), but we shall not do it. For us a manifold is always smooth (in the sense of $C^\infty$), therefore we shall often drop the adjective.

**Example 1.10.** Consider $M = \mathbb{R}^2$. Introduce two charts: $(U_1, \varphi_1), (U_2, \varphi_2)$, where open set $U_1$ is $\mathbb{R}^2$ itself, $U_1 = \mathbb{R}^2$, and open set $U_2$ is $\mathbb{R}^2$ without points of the left ray of $x$ axis: $U_2 = \mathbb{R}^2 \backslash I_-$, where $I_- = \{(x, y), y = 0, x \leq 0\}$. Take $\varphi_1 = \mathbf{id}$ and $\varphi_2 \colon (x, y) \mapsto (r, t)$ are polar coordinates. These two charts form an atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$. Transition functions are defined on open sets and they are equal to

$$\begin{aligned} \Psi_{12}(r, t) = \varphi_1 \varphi_2^{-1} \colon (r, t) &\mapsto (x = r\cos t, y = r \sin t) \\ \Psi_{21}(x, y) = \varphi_2 \varphi_1^{-1} \colon (x, y) &\mapsto (r = \sqrt{x^2 + y^2}, t = \arctan \tfrac{y}{x}) \end{aligned} \tag{1.2}$$

**Example 1.11.** Consider $S^1$ as the set of the point $x^2 + y^2 = 1$ on $\mathbb{R}^2$. Introduce two charts: $\varphi_1$ on $U_1 = S^1 \backslash N$ and $\varphi_2$ on $U_2 = S^1 \backslash S$, where $N = (0, 1)$ is North Pole, $S = (0, -1)$ is the South pole, and

$$\varphi_1(x, y) = u = \frac{2x}{1 - y}, \qquad \varphi_2(x, y) = u' = \frac{2x}{1 + y}$$

(see Example 1.4) are stereographic projections. They correspond to the stereographic projections from the 'north pole' $N = (0, 1)$ and the 'south pole' $S = (0, -1)$ respectively. For the change of coordinates we obtain, that $uu' = 1$

$$u' = \Psi_{21}(u) = \varphi_2 \varphi_1^{-1}(u) = \frac{1}{u},$$

Therefore it is smooth, and we conclude that $S^1$ with this atlas is a smooth manifold of dimension 1. (See also Homeworks 1 and 2)

In the same way we can obtain a smooth atlas consisting of two charts on any sphere $S^n \subset \mathbb{R}^{n+1}$. (See Homeworks 1,2) This makes $S^n$ a smooth manifold of dimension $n$.

**Example 1.12.** A point of $\mathbb{R}P^n$ can be identified with a non-zero vector $\mathbf{v} = \mathbb{R}^{n+1}$ considered up to a non-zero scalar factor, $\mathbf{v} \sim k\mathbf{v}$, $k \neq 0$. The coordinates of $\mathbf{v}$ considered up to a factor are written as $[x^1 : \ldots : x^n : x^{n+1}]$ and traditionally called the *homogeneous coordinates* on $\mathbb{R}P^n$. (They are not coordinates in the true sense, because are defined only up to a factor.) The construction described in Example 1.8 gives a chart $\varphi \colon U_{n+1} \to \mathbb{R}^n$,

$$[y^1 : y^2 : \cdots : y^n : 1] \mapsto (y^1, \ldots, y^n)$$

where $U_{n+1}$ is the set of points $[y^1 : y^2 : \cdots : y^n : y^{n+1}]$ in $\mathbb{R}P^n$ such that $y^{n+1} \neq 0$. The image of $U$ is the whole $\mathbb{R}^n$.

Similarly we can define other charts $(U_k, \varphi_{(k)}$, $k = 1, \ldots, n$, where $U_k$ is the set of points $[y^1 : y^2 : \cdots : y^n : y^{n+1}]$ in $\mathbb{R}P^n$ such that $y^k \neq 0$. Together with the chart $(U_{n+1}, \varphi_{(n+1)})$ they make an atlas for $\mathbb{R}P^n$ consisting of $n+1$ charts. It is smooth. Hence, $\mathbb{R}P^n$ with this atlas becomes an $n$-dimensional smooth manifold. Coordinates in any of these charts are traditionally called the *inhomogeneous coordinates* on $\mathbb{R}P^n$. (See also the Homework 1,2)

**Example 1.13.** Acting similarly for $\mathbb{C}P^n$, we obtain the $n+1$ charts

$$\varphi_{(k)} \colon \mathbb{C}^n \to \mathbb{C}P^n$$

again giving a smooth atlas. Hence $\mathbb{C}P^n$ has the structure of a $2n$-dimensional manifold. (Each complex coordinate gives two real coordinates.)

Before moving forward, the following remark should be made. Suppose on the same set $M$, two smooth atlases are defined, so we have two smooth manifolds, $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$, where we used script letters $\mathcal{A}_1$ and $\mathcal{A}_2$ for denoting atlases. For example, for the circle $S^1$ (or the sphere $S^n$) we can consider the atlas consisting of the two 'stereographic' charts as above or an atlas constructed using angular coordinates. Do they define "the same" circle as a smooth manifold, or not? We obviously have to introduce some equivalence relation between atlases. This goes as follows. Two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on the same set $M$ are *equivalent* if their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. This is equivalent to saying that all changes of coordinates between charts from $\mathcal{A}_1$ and charts from $\mathcal{A}_2$ are smooth. (Clearly, this holds for the those two atlases for $S^1$ or $S^n$.) We may, therefore, amend slightly our Definition 1.2 by saying that *a manifold is a pair consisting of a set and an equivalence class of smooth atlases.* (Practically, we shall always work with some particular atlas from this class.)

**Example 1.14.** Numerous 'natural' ways of introducing a manifold structure on the sphere $S^n$, namely, by using angular coordinates; by using the stereographic projection; see also Example 1.25, — all give equivalent atlases, as one can check. Therefore we may unambiguously speak of $S^n$ as of a smooth manifold.

## 1.3 Smooth functions and smooth maps

Consider a manifold $M$. In the sequel we assume for each manifold a particular smooth atlas is chosen, and when we speak of charts, we mean charts from that atlas.

Let $f$ be a function on the smooth manifold $M$. If $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$ is an atlas on $M$ then one can consider for any chart $\varphi_\alpha \colon U_\alpha \to V_\alpha \subset \mathbb{R}^n$ the function

$$f_\alpha = f \circ \varphi_\alpha^{-1} \colon (x^1_{(\alpha)}, \dots x^n_{(\alpha)}) \to \mathbb{R} \tag{1.3}$$

These local functions are functions take number value and are defined on $\mathbb{R}^n$—these functions we study in the course of many variables.

**Definition 1.3.** A function $f \colon M \to \mathbb{R}$ is called *smooth* (or *differentiable*, note remarks made above) if for all charts $\varphi_\alpha \colon U_\alpha \to V_\alpha \subset \mathbb{R}^n$ the compositions $f_\alpha = f \circ \varphi_\alpha^{-1} \colon U_\alpha \to \mathbb{R}$ are smooth functions on open sets $V_\alpha \subset \mathbb{R}^n$. The set of all smooth functions on $M$ is denoted $C^\infty(M)$.

Simply speaking, if we describe points of $M$ by their coordinates (w.r.t. a particular chart), then smooth functions on $M$ are expressed as smooth function of coordinates. The requirement that all changes of coordinates $\varphi_\alpha^{-1} \circ \varphi_\beta$ are smooth, can be seen as a compatibility condition: if a function on $M$ is smooth in one coordinate system, it will be smooth also in any other coordinate system. (The composition of smooth functions defined on open domains of Euclidean spaces is smooth, as it follows from the chain rule.)

In a similar way we can define smooth maps between manifolds. Consider manifolds $M_1^{n_1}$ and $M_2^{n_2}$. We shall denote charts on $M_1^{n_1}$ and $M_2^{n_2}$ by adding subscripts, such as $\varphi_{1\alpha} \colon U_{1\alpha} \to M_1$, etc. Let $F \colon M_1 \to M_2$ be a map. Consider the subset $F^{-1}(U_{2\mu}) \cap U_{1\alpha}$ and assume that it is not empty. Then $F$ maps it to $U_{2\mu}$; on the other hand, consider its image $\varphi_{1\alpha}\left(F^{-1}(U_{2\mu}) \cap U_{1\alpha}\right)$. We have a map

$$\varphi_{2\mu} \circ F \circ \varphi_{1\alpha}^{-1} \colon \ \varphi_{1\alpha}\left(F^{-1}(U_{2\mu}) \cap U_{1\alpha}\right) \to V_{2\mu}\,.$$

**Definition 1.4.** A map $F\colon M_1 \to M_2$ is *smooth* if all sets $\varphi_{1\alpha}\left(F^{-1}(U_{2\mu}) \cap U_{1\alpha}\right)$ are open and the above map $\varphi_{2\mu}^{-1} \circ F \circ \varphi_{1\alpha}$ is a smooth map (from an open set of $\mathbb{R}^{n_1}$ to an open set in $\mathbb{R}^{n_2}$), for all indices $\alpha, \mu$. The set of all smooth maps from $M_1$ to $M_2$ is denoted $C^\infty(M_1, M_2)$.

The same as for functions $f\colon M \to \mathbb{R}$, a map $F\colon M_1 \to M_2$ is smooth, simply speaking, if it is smooth when expressed in coordinates (using arbitrary coordinate systems on both manifolds).

**Theorem 1.1.** *The composition of smooth maps of manifolds is smooth. The identity map, for any manifold, is smooth.*

Hence we have a **category** of smooth manifolds (see Appendix).
Isomorphisms in this category are called diffeomorphisms.

**Definition 1.5.** A map of manifolds $F\colon M_1 \to M_2$ is a *diffeomorphism* if it is smooth, invertible, and the inverse map $F^{-1}\colon M_2 \to M_1$ is also smooth. Manifolds $M_1$ and $M_2$ are called *diffeomorphic* if there is a diffeomorphism $F\colon M_1 \to M_2$. Notation: $M_1 \cong M_2$.

**Example 1.15.** $\mathbb{R}_+ \cong \mathbb{R}$. Use the maps exp and log.

**Example 1.16. (Counterexample). Consider the bijection $x \mapsto x^3$, $\mathbb{R} \to \mathbb{R}$. It is not diffeomorphism (see Homework 2).**

Let us consider elementary properties of smooth functions.

**Theorem 1.2.** *Consider a manifold $M^n$. All constants are smooth functions. The sum and product of smooth functions are smooth functions.*

*Proof.* Consider, for example, the sum of two functions $f, g \in C^\infty(M)$. We need to check that $f + g$ also belongs to $C^\infty(M)$. By definition that means that for any chart $\varphi\colon V \to M$ (where $V \subset \mathbb{R}^n$), the composition $(f + g) \circ \varphi$ belongs to $C^\infty(V)$. We have $((f + g) \circ \varphi)(\mathbf{x}) = (f + g)(\varphi(\mathbf{x})) = f(\varphi(\mathbf{x})) + g(\varphi(\mathbf{x})) = (f \circ \varphi)(\mathbf{x}) + (g \circ \varphi)(\mathbf{x}) = (f \circ \varphi + g \circ \varphi)(\mathbf{x})$ for all $\mathbf{x} \in V$. Therefore $(f + g) \circ \varphi = f \circ \varphi + g \circ \varphi \in C^\infty(V)$, since the sum of two smooth function of $n$ variables is a smooth function. $\square$

Recall that an *algebra* over any field is a vector space which is also a ring, such that multiplication is a linear operation, i.e. $(ka)b = k(ab)$ for all algebra elements $a, b$ and field elements $k$.

Theorem 1.2 means that $C^\infty(M)$ is an algebra over the field of real numbers $\mathbb{R}$. The algebra $C^\infty(M)$ is associative and commutative, and there is a unit 1.

Elements of $C^\infty(M^n)$ have **local** appearance as smooth functions of $n$ real variables (that is how they are defined).

**Example 1.17.** Elements of the algebra $C^\infty(S^1)$ can be locally written as smooth functions $f = f(\varphi)$ of the polar angle $\varphi \in (0, 2\pi)$ or as smooth functions of the variable $u = u_N$ introduced above.

**Example 1.18.** Similarly, functions on $S^2$ locally look as functions of the variables $u^1, u^2$ defined by a stereographic projection, or as functions of the angles $\theta, \varphi$, (or any other local coordinates that can be introduced on the sphere).

Recall that a *homomorphism* of algebras is a linear map (of algebras regarded as vector spaces) preserving products: $\psi(ab) = \psi(a)\psi(b)$. An *isomorphism* of algebras is an algebra homomorphism which is invertible.

Since, in general, coordinates on a manifold are defined only locally (there is no coordinate chart covering the whole manifold), the algebra $C^\infty(M^n)$ for an $n$-dimensional manifold and the algebra $C^\infty(\mathbb{R}^n)$ are **not** isomorphic in spite of their elements having the same 'local' appearance.

Let $X$ and $Y$ be arbitrary sets, $F\colon X \to Y$, an arbitrary map of sets. Denote by $\mathrm{Fun}(X)$ and $\mathrm{Fun}(Y)$ the sets of all functions $X \to \mathbb{R}$ and $Y \to \mathbb{R}$, respectively. Obviously, they are algebras. Consider $g \in \mathrm{Fun}(Y)$. The operation $g \mapsto g \circ F$ is a map $\mathrm{Fun}(Y) \to \mathrm{Fun}(X)$.

**Definition 1.6.** The operation $g \mapsto g \circ F$ is called the *pullback* $F$. Notation: $F^*(g) = g \circ F$.

The pullback $F^*$ is a map $\mathrm{Fun}(Y) \to \mathrm{Fun}(X)$ (in the opposite direction to $F\colon X \to Y$!).

**Lemma 1.1.**

$$(F \circ G)^* = G^* \circ F^* \tag{1.4}$$

*Proof.* Left as exercise. □

Notice the opposite order in the RHS of formula (1.4).

**Theorem 1.3.** *Suppose $F\colon M_1 \to M_2$ is a smooth map of manifolds. Then the pullback is an algebra homomorphism $F^*\colon C^\infty(M_2) \to C^\infty(M_1)$.*

*Proof.* We have to check that the pull-back of a smooth function on $M_2$ under a smooth map $F\colon M_1 \to M_2$ is a smooth function on $M_1$. Indeed, by the definition, for $f \in C^\infty(M_2)$, we have $F^*f = f \circ F$, which as smooth as the composition of smooth maps. Now we need to check that $F^*$ is a homomorphism, i.e., it is linear and maps products to products. It is the same as the proof that $C^\infty(M)$ for a given $M$ is an algebra. For example, $\big(F^*(fg)\big)(\mathbf{x}) = ((fg) \circ F)(\mathbf{x}) = (fg)(F(\mathbf{x}) = f(F(\mathbf{x}))g(F(\mathbf{x})) = \big((f \circ F)(g \circ F)\big)(\mathbf{x}) = \big(F^*f\, F^*g\big)(\mathbf{x})$ for all $\mathbf{x} \in M_1$. In other words, $F^*(fg) = F^*f \cdot F^*g$. $\square$

**Example 1.19.** Consider a map $F\colon \mathbb{R} \to S^1$ that sends $t \in \mathbb{R}$ to $F(t) = (\cos t, \sin t) \in S^1 \subset \mathbb{R}^2$. It is smooth. Because it is onto, the pullback is injective. (Check!) It follows that $C^\infty(S^1)$ can be identified with its image in $C^\infty(\mathbb{R})$, which is the subalgebra consisting of all $2\pi$-periodic functions.

## 1.4 Constructions of manifolds.

*Products.*

Let $X$ and $Y$ are manifolds of dimensions $n$ and $m$, respectively. Then the Cartesian product $X \times Y$ has a natural structure of a manifold of dimension $n + m$.

Construction: if $\varphi_{1\alpha}\colon U_{1\alpha} \to V_{1\alpha}$ are charts for $X$ and $\varphi_{2\mu}\colon U_{2\mu} \to V_{2\mu}$ are charts for $Y$ (here $V_{1\alpha} \subset \mathbb{R}^n$, $V_{2\mu} \subset \mathbb{R}^m$), then $U_{1\alpha} \times U_{2\alpha}$ are subsets in $X \times Y$ and

$$\varphi_{1\alpha} \times \varphi_{2\mu}\colon U_{1\alpha} \times U_{2\mu} \to V_{1\alpha} \times V_{2\mu}$$

are charts for $X \times Y$. Here

$$\begin{aligned} \text{for}(\mathbf{x},\mathbf{y}) \in \mathrm{X} \times \mathrm{Y} \quad \varphi_{1\alpha} \times \varphi_{2\mu}\colon \ (\mathbf{x},\mathbf{y}) &\mapsto (x^1, \ldots, x^n, y^1, \ldots, y^m)\,, \\ \varphi_{1\alpha}(\mathbf{x}) = (x^1, \ldots, x^n), &\quad \varphi_{2\mu}(\mathbf{y}) = (y^1, \ldots, y^m)\,. \end{aligned}$$

If $\{(V_{1\alpha}, \varphi_{1\alpha})\}$ and $\{(V_{2\mu}, \varphi_{2\mu})\}$ are smooth atlases, then a direct check shows that the atlas $\{(V_{1\alpha} \times V_{2\mu}, \varphi_{1\alpha} \times \varphi_{2\mu})\}$ is also smooth.

**Example 1.20.** The product $S^1 \times S^1$ is called the 2-*torus*, and is diffeomorphic with the "surface of a bagel" in $\mathbb{R}^3$.

**Example 1.21.** For any $n$, the $n$-fold product $S^1 \times \ldots \times S^1$ is called the $n$-*torus*. Notation: $T^n$.

**Example 1.22.** For any manifold $M$, the product $M \times \mathbb{R}$ is called an (infinite) *cylinder* over $M$.

**Example 1.23.** If we replace $\mathbb{R}$ by the open interval $(0,1)$, we obtain a finite cylinder $M \times (0,1)$. (Sketch a picture for $M = S^1$.)

**Remark 1.2.** The product of a manifold with a closed segment $[0,1]$ will not be a manifold. (Why?) It gives an example of a "manifold with boundary", a notion to be discussed later.

*Specifying manifolds by equations*

First of all, we need to recall what 'independent equations' means.

**Example 1.24.** Let $k \leq N$. Consider a linear system

$$A\mathbf{y} = 0 \tag{1.5}$$

where $\mathbf{y} \in \mathbb{R}^N$ and $A$ is an $k \times N$ matrix. Equations (1.5) are called *independent* if the $k$ rows of the matrix $A$ are linearly independent. As we know, in this case the set of solutions of (1.5) is a vector space of dimension $N-k$. If we solve this system by Gauss elimination, we arrive at some $N-k$ variables that are 'free' (can be set to any values), and the other $k$ are expressed as linear functions of them, and this gives the general solution.

This example is a linear model of the general situation.

Consider in $\mathbb{R}^N$ a system of $k$ equations

$$\begin{cases} F^1(y^1, \dots, y^N) = 0 \\ \qquad\qquad \dots \\ F^k(y^1, \dots, y^N) = 0 \end{cases} \tag{1.6}$$

where the LHS's are smooth functions. Denote by $S \subset \mathbb{R}^N$ the set of solutions of (1.6), i.e., the set of all points of $\mathbb{R}^N$ satisfying equations (1.6). We say that equations (1.6) are *independent* if the rows of the matrix of partial derivatives

$$\frac{\partial \mathbf{F}}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial F^1}{\partial y^1} & \frac{\partial F^1}{\partial y^2} & \dots & \frac{\partial F^1}{\partial y^N} \\ \dots & & & \\ \frac{\partial F^k}{\partial y^1} & \frac{\partial F^k}{\partial y^2} & \dots & \frac{\partial F^k}{\partial y^N} \end{pmatrix} \tag{1.7}$$

are linearly independent (at least at the points belonging to the set of solutions $S \subset \mathbb{R}^N$).

**Theorem 1.4.** *If equations* (1.6) *are independent, then the set of solutions $S$ has a natural structure of a smooth manifold of dimension $N-k$.*

*Proof.* Suppose the set of solutions $S$ is non-empty. (If empty, nothing to prove: empty set is a manifold.) For each $\mathbf{y}_0 \in S$, we can apply the implicit function theorem: near $\mathbf{y}_0$, for a solution of the system (1.6), from the $N$ coordinates $y^1, \dots, y^N$ one can choose some $N-k$ coordinates as independent variables and express the rest as functions of them. For concreteness, let it be the last $N-k$ coordinates: there are smooth functions $\varphi^1, \dots \varphi^k$ defined in a neighborhood of $(y_0^{k+1}, \dots, y_0^N) \in \mathbb{R}^{N-k}$ so that the points $\mathbf{y}$ of the form

$$\mathbf{y} = \big(\varphi_1(y^{k+1}, \dots, y^N), \dots, \varphi_1(y^{k+1}, \dots, y^N), y^{k+1}, \dots, y^N\big) \in \mathbb{R}^N$$

satisfy the system (1.6) for all values of $y^{k+1}, \dots, y^N$ in the said neighborhood. Rename $y^{k+1}, \dots, y^N$ as $u^1, \dots, u^{N-k}$. The map $(u^1, \dots, u^{N-k}) \mapsto \big(\varphi_1(u^1, \dots, u^{N-k}), \dots, \varphi_1(u^1, \dots, u^{N-k}), u^1, \dots, u^{N-}$ is a chart for the set $S$. The inverse map is simply given by the projection on the last $N-k$ coordinates: $(y^1, \dots, y^N) \mapsto (u^1 = y^{k+1}, \dots, u^{N-k} = y^N)$. There is such a chart defined for each point in $S$. Therefore their collection gives an atlas for $S$. We immediately see that when a point $\mathbf{y} \in S$ can be covered by two charts, the change of coordinates between them will be smooth (as composition of smooth maps). Hence $S$ is a smooth manifold. $\square$

We often use the special case of this Theorem when the set is defined by one equation $F = 0$ in the space $\mathbb{R}^N$. The condition (1.8) in this case means that the vector

$$\left(\frac{\partial F}{\partial y^1}, \frac{\partial F}{\partial y^2}, \dots, \frac{\partial F}{\partial y^N}\right)$$

is not equal to zero at least at the points of $S$.

If this condition is satisfied then the set $S$ is the manifold of the dimension $N-1$.

**Example 1.25.** Consider once again the sphere $S^2$ (defined as the unit sphere in $\mathbb{R}^3$). It is specified by the equation

$$x^2 + y^2 + z^2 = 1\,,$$

where we used traditional notation for coordinates on $\mathbb{R}^3$. In other words, we have a "system" consisting of one equation

$$F(x, y, z) = 0$$

where $F(x, y, z) = x^2 + y^2 + z^2 - 1$. The set of solutions $S$ is our sphere $S^2$. The matrix of partial derivatives is the row matrix

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x, 2y, 2z)\,.$$

We see that this row vector never vanishes for $(x, y, z) \neq 0$, in particular, it does not vanish for all $(x, y, z) \in S$. Therefore our "system" (consisting of a single equation) is independent and Theorem 1.4 applies. Once again we obtain the manifold structure of $S^2$. (The main point in Theorem 1.4 is the possibility to solve the system in question w.r.t. some of the variables so that the remaining variables are 'free' and can be taken as coordinates, thus giving a manifold structure. This is deduced from the implicit function theorem. For the equation of the sphere we, of course, can resolve it w.r.t. to one the coordinates directly.)

**Counterexample** Consider the set $S$ in $\mathbf{E}^3$ defined by the equation

$$x^2 + y^2 + z^2 = 0$$

The vector $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x, 2y, 2z)$ is equal to zero if $x^2 + y^2 + z^2 = 0$. So the conditions of the Theorem are not satisfied.

The set $S$ is the point $x = y = z = 0$. It is not 2-dimensional manifold.

One can see other examples in Homeworks 2,3.

In practical situations it often happens that interesting objects are defined by equations that are not independent, and constructing an equivalent system of independent equations may be awkward, if at all possible. The following generalisation of the previous theorem is helpful (we skip a proof):

**Theorem 1.5 (Manifolds specified by equations of constant rank).** *If a system as above is not necessarily independent, but has a constant rank $r$, then its set of solutions is a manifold of dimension $N - r$.*

Let us explain what is meant by 'constant rank' here. We say that a system of equations (1.6) has *constant rank* if at all points of the set of solutions $S$, the rank of the matrix of partial derivatives (1.7) is a constant number $r \leq k$ (the same for all points $\mathbf{y} \in S$). Recall that the *rank* of a matrix is the dimension of the vector space spanned by its rows (the row space) or the dimension of the vector space spanned by its columns (the column space). These two numbers coincide. (A particular case is of course $r = k$, i.e., the case of independent equations treated in Theorem 1.4.)

How do we practically know that the rank is constant? We should look at the vector space of solutions of the following auxiliary linear system:

$$\frac{\partial F^\mu}{\partial y^a}(\mathbf{y})\dot{y}^a = 0\,. \tag{1.8}$$

Here $\mathbf{y}$ is a given point in the set of solutions. The unknowns are variables that we denoted by $\dot{y}^a$. The matrix of the system is exactly the matrix of partial derivatives for which we want to know whether its rank is constant (independent of $\mathbf{y}$), or not. From linear algebra we know that the dimension of the space of solutions of (1.8) is the number of variables (which is $N$) minus the rank of the matrix (which is $r$). Therefore the rank $r$ of the matrix

$\left(\frac{\partial F^\mu}{\partial y^a}(\mathbf{y})\right)$ is constant for $\mathbf{y} \in S$ if and only if the dimension of the space of solutions of (1.8) does not depend on $\mathbf{y} \in S$.

Notice that the linear system (1.8) can be obtained by the formal differentiation of the system (1.6) w.r.t. a parameter $t$ ("time"), so that 'dot' stands for the "time derivative". (Later we shall see the geometrical meaning of this.)

**Example 1.26.** Recall that a matrix $A$ is called orthogonal is $AA^T = E$, where $E$ is the identity matrix. The set of all real orthogonal $n \times n$ matrices is denoted $O(n)$. One can check that $O(n)$ is a group, called the *orthogonal group*. We claim that $O(n)$ has a natural manifold structure. Let us apply Theorem 1.5. Notice that a single matrix equation such as $AA^T = E$ is in fact a system of $n^2$ equations for the matrix entries. To see that it has constant rank we consider $A$ as a function of a parameter $t$, $A = A(t)$, and differentiate the equation w.r.t. $t$. We obtain the linear system $\dot{A}A^T + A^T\dot{A} = 0$ or $(\dot{A}A^T)^T = -\dot{A}A^T$. (Here $A \in O(n)$ and the unknown is $\dot{A}$.) The space of solutions is therefore isomorphic to the space of all skew-symmetric $n\times n$ matrices. Hence it has constant dimension $n(n-1)/2$ independent of $A$ and the rank of the original system is constant. The dimension of the manifold is $n(n-1)/2$. (The dimension of the space of all skew-symmetric $n \times n$ matrices can be counted as the number of independent entries for such a matrix: $\frac{1}{2}(n^2 - n)$, where $n$ is the number of the diagonal entries, which all have to be zero.)

**Example 1.27.** A particular simple case is the group $O(2)$ of the orthogonal $2\times2$ matrices. The equation $AA^T = E$ for $n = 2$ can be solved explicitly (do this!). Any orthogonal $2 \times 2$ matrix is either

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

or

$$A = \begin{pmatrix} -\cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

(Geometrically it is either the rotation through some angle $\alpha$ or a rotation followed by the inversion in a coordinate axis.) As expected we obtain an example of a one-dimensional manifold: $2(2-1)/2 = 1$. Moreover, we see that $O(2) \cong S^1 \cup S^1$ (the disjoint union of two circles).

Manifolds obtained as above are called *surfaces of codimension $r$ (or dimension $N-r$)* in $\mathbb{R}^N$. Actually, any manifold is a "multidimensional surface" in this sense, as stated in the following theorem.

We defined manifolds in abstract way. In this susbsection we showed that some manifolds can be realised as subsets in $R^n$.

In fact there is a very deep theorem

**Theorem 1.6.** *Any manifold $M^n$ can be embedded into a Euclidean space $\mathbb{R}^N$ for some $N$.*

Proof to be discussed later. Two points: 1) there is no 'standard' or 'canonical' way (a prescription good for all manifolds) of embedding manifolds to $\mathbb{R}^N$, and the dimension $N$ depends on how we arrange an embedding (we shall discuss minimizing $N$ later); 2) knowing that a particular manifold is a surface in $\mathbb{R}^N$ does not usually help and is a rather superfluous information. Practically it is more convenient to work intrinsically (in terms on a manifold itself) rather then in terms of an ambient $\mathbb{R}^N$.

## 1.5 Appendix. The notion of a category

A *category* is an algebraic structure (generalizing groups and semigroups) consisting of the following data: a set whose elements are called *objects* and a set whose elements are called *arrows* or *morphisms*, so that for each arrow there are two uniquely defined objects called its *source* and *target* (it is said that an arrow 'goes' from its source to its target); there is a binary operation called the *product* or *composition* of arrows, defined for any two arrows $a_1$ and $a_2$ if the target of $a_2$ coincides with the source of $a_1$; then the source of the composition $a_1 \circ a_2$ is equal to the source of $a_2$ and the target of $a_1 \circ a_2$, to that of $a_1$. Two properties are satisfied: composition is associative and for each object $X$ there is a morphism $1_X$, called the *identity* for $X$, such that $a \circ 1_X = a$ and $1_X \circ a = a$ for all arrows $a$ going from, and to $X$ respectively.

(Categorical constructions are best understood by drawing diagrams where arrows are represented by actual arrows joining letters denoting the corresponding sources and targets.)

One obtains a group if there is only one object and all arrows are assumed to be invertible, i.e., for each $a$ there is $a^{-1}$ such that $a \circ a^{-1} = 1$ and $a^{-1} \circ a = 1$, where 1 stands for the identity corresponding to the single object. The arrows are the group elements in this case. If no invertibility of arrows is assumed, a category with a single object is what is known as a *monoid* (or a 'semigroup with identity').

In the same way as groups appear as transformation groups, i.e., families of certain invertible transformations of a given object (a set, a vector space, etc.), semigroups or monoids appear as families of not necessarily invertible transformations of a fixed object. Categories can appear as families of transformations between different objects. For example, for a fixed vector space $V$, all invertible linear operators $A\colon V \to V$ make a group (called the general linear group of $V$); all linear operators $A\colon V \to V$, not necessarily invertible, make a monoid, where the identity is the identity operator $1_V\colon V \to V$; finally, the collection of arbitrary linear transformations $A\colon V \to W$ between all vector spaces make a category. The 'objects' for this category are the vector spaces and the 'arrows', the linear maps. The 'source' and the 'target' of a map have their natural meanings. The composition is the composition of maps in the natural sense. This category is called the *category of vector spaces.*

Similarly appear such examples of categories as the *category of sets* (objects are sets, morphisms are arbitrary maps of sets), the *category of topological spaces* (objects are topological spaces, morphisms are continuous maps), the *category of groups* (objects are groups, morphisms are group homomorphisms), the *category of associative rings* (objects are associative rings, morphisms are ring homomorphisms), etc. From the viewpoint of category theory, in all these examples 'objects' are treated as whole entities without internal structure (which was used only for specifying the set of arrows); it is morphisms that play a role. One should also note that, the same as 'abstract' groups not arising as transformation groups, there are 'abstract' categories where arrows are not defined as maps. An example is the *pair category* of a given set $S$ where the objects are defined as the elements of $S$ and the arrows are defined as the pairs $(X,Y) \in S \times S$, with the composition law $(X,Y) \circ (Y,Z) = (X,Z)$.

Category theory plays a very important role in modern mathematics, primarily as a unifying language for many algebraic and geometric constructions.

A simplest "abstract" notion from category theory is that of an **isomorphism**. A morphism $a$ in a given category is called an *isomorphism* if there is a morphism $b$ going in the opposite direction such that $a \circ b$ and $b \circ a$ are the identities for the respective objects. It follows that in this case $b$ is defined uniquely (check from the axioms of a category!). It is called the *inverse* of $a$ and denoted $a^{-1}$. Two objects $X$ and $Y$ are called *isomorphic* if there is at least one isomorphism going from $X$ to $Y$. One can see that this defines an equivalence relation on the set of objects (check!), also called 'isomorphism'. Examples: in the category of groups, the isomorphisms are the isomorphisms of groups; in the category of sets, the isomorphisms are the bijections; in the category of topological spaces, the isomorphisms are the homeomorphisms. In the category of smooth manifolds, the isomorphisms are the diffeomorphisms.

# 2 Tangent vectors and related objects

## 2.1 Tangent vectors

### 2.1.1 Velocity vectors

Consider a smooth manifold $M = M^n$. A *(smooth) curve* in $M$ is a smooth map $\gamma\colon (\alpha, \beta) \to M$. In other words, we have $t \mapsto \mathbf{x}(t) \in M$. How is it possible to define the velocity of $\gamma$?

Let us fist examine the case of $M = \mathbb{R}^n$. Then in a straightforward way we can consider

$$\frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t}\,.$$

The top of the fraction is a vector in $\mathbb{R}^n$ (as the difference of two points), the bottom is a number; hence the whole thing is a vector in $\mathbb{R}^n$. It makes sense to pass to the limit when $\Delta t \to 0$ and, assuming the limit exists, we obtain the vector

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} := \lim_{\Delta \to 0} \frac{\mathbf{x}(t + \Delta t) - \mathbf{x}(t)}{\Delta t},$$

which, by definition, is the *velocity* of our curve. Unfortunately, we cannot do the same for an arbitrary manifold $M$, because taking difference of two points will not make sense. The only tool we have for manifolds are coordinates. Therefore we have to examine the case of $\mathbb{R}^n$ further and see how velocity can be expressed using coordinates.

Writing $\mathbf{x} = (x^1, \ldots, x^n)$ and working out the difference $\mathbf{x}(t+\Delta t) - \mathbf{x}(t)$, we arrive at $\big(x^1(t+\Delta t) - x^1(t), \ldots, x^n(t+\Delta t) - x^n(t)\big)$. Therefore calculating the velocity amounts to taking the time derivative of each coordinate, and the velocity vector of a curve in $\mathbb{R}^n$ has the coordinate expression

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = (\dot{x}^1, \ldots, \dot{x}^n)$$

in standard coordinates. Introducing the vectors $\mathbf{e}_i$=(0,...,1...,0) of the standard basis, with 1 in the $i$-th position and 0 everywhere else, we can re-write this also as

$$\dot{\mathbf{x}} = \dot{x}^1 \mathbf{e}_1 + \ldots \dot{x}^n \mathbf{e}_n \,.$$

**Example 2.1.** For a curve in $\mathbb{R}^3$ we have

$$\dot{\mathbf{x}} = (\dot{x}, \dot{y}, \dot{z}) = \dot{x}\mathbf{e}_1 + \dot{y}\mathbf{e}_2 + \dot{z}\mathbf{e}_3$$

using the traditional notation for coordinates, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$. In $\mathbb{R}^2$ we have

$$\dot{\mathbf{x}} = (\dot{x}, \dot{y}) = \dot{x}\mathbf{e}_1 + \dot{y}\mathbf{e}_2 \,.$$

To progress with our goal of defining velocity on arbitrary manifolds, we need to find an expression of velocity vector for $\mathbb{R}^n$ in coordinate systems that are not necessarily standard coordinates used above.

**Example 2.2.** Let us work out the expression for velocity of a curve in $\mathbb{R}^2$ in polar coordinates $r, \theta$ rather than Cartesian coordinates $x, y$. Suppose a curve $\mathbf{x} = \mathbf{x}(t)$ is defined using polar coordinates as $r = r(t)$, $\theta = \theta(t)$. To calculate the velocity $\dot{\mathbf{x}}$ we use the chain rule: a point $\mathbf{x} \in \mathbb{R}^2$ is considered as a composite function of the time, first $\mathbf{x} = \mathbf{x}(r, \theta)$ and then $r = r(t)$, $\theta = \theta(t)$. We obtain

$$\dot{\mathbf{x}} = \dot{r}\,\frac{\partial \mathbf{x}}{\partial r} + \dot{\theta}\,\frac{\partial \mathbf{x}}{\partial \theta} \tag{2.1}$$

where partial derivatives w.r.t. $r$ and $\theta$ are vectors, and $\dot{r}$, $\dot{\theta}$ are scalar coefficients. If we introduce the vectors

$$\mathbf{e}_r := \frac{\partial \mathbf{x}}{\partial r} \quad \text{and} \quad \mathbf{e}_\theta := \frac{\partial \mathbf{x}}{\partial \theta}\,, \tag{2.2}$$

we may write

$$\dot{\mathbf{x}} = \dot{r}\,\mathbf{e}_r + \dot{\theta}\,\mathbf{e}_\theta\,,$$

very similarly to the expression in Cartesian coordinates.

Sometimes for vectors $\mathbf{e}_r, \mathbf{e}_t$ we use notations $\frac{\partial}{\partial r}$ instead $\frac{\partial \mathbf{x}}{\partial r}$ and respectively $\frac{\partial}{\partial \theta}$ instead $\frac{\partial \mathbf{x}}{\partial \theta}$.

Note that the vectors $\mathbf{e}_r, \mathbf{e}_\theta$ associated with polar coordinates on $\mathbb{R}^2$ depend on a point $\mathbf{x} \in \mathbb{R}^2$ and make a basis for each $\mathbf{x}$ except for the origin, where $\mathbf{e}_r$ is not defined and $\mathbf{e}_\theta$ vanishes. (Indeed, one can find $\mathbf{e}_r = (\cos\theta, \sin\theta), \mathbf{e}_\theta = (-r\sin\theta, r\cos\theta)$ in standard coordinates.) Hence the expression $\dot{\mathbf{x}} = \dot{r}\,\mathbf{e}_r + \dot{\theta}\,\mathbf{e}_\theta$ is nothing but the expansion over this basis (at a given point $\mathbf{x}$). On the other hand, the vectors of the standard basis in $\mathbb{R}^2$ can be also written as partial derivatives w.r.t. the coordinates:

$$\mathbf{e}_1 = (1,0) = \frac{\partial}{\partial x}(x,y) = \frac{\partial \mathbf{x}}{\partial x}, \quad \text{and} \quad \mathbf{e}_2 = (0,1) = \frac{\partial}{\partial y}(x,y) = \frac{\partial \mathbf{x}}{\partial y}\,.$$

Generalising to $\mathbb{R}^n$, we conclude that for arbitrary "curvilinear" coordinates $y^1, \ldots, y^n$ (we use a different letter to distinguish from standard coordinates denoted above as $x^1, \ldots, x^n$) there is an associated basis of vectors

$$\mathbf{e}_1 := \frac{\partial \mathbf{x}}{\partial y^1}\,,\ \ldots,\ \mathbf{e}_n := \frac{\partial \mathbf{x}}{\partial y^n}$$

(although we use the same letters $\mathbf{e}_i$, these vectors are not to be confused with the standard basis vectors above), depending on a point,[2] with respect to which the velocity of any curve in $\mathbb{R}^n$ has the expansion

$$\dot{\mathbf{x}} = \dot{y}^1\,\mathbf{e}_1 + \ldots + \dot{y}^n\,\mathbf{e}_n\,.$$

In other words, we may say that in an **arbitrary** coordinate system on $\mathbb{R}^n$, the array of the time derivatives of the coordinates

$$(\dot{y}^1, \ldots, \dot{y}^n)$$

[2]The linear independence of the vectors $\mathbf{e}_i$ for each $\mathbf{x}$ is, in fact, a part of the definition of what is a 'system of curvilinear coordinates' on a domain of $\mathbb{R}^n$, the other principal condition being the that the correspondence between points and coordinates is one-to-one.

gives the components of the velocity $\dot{\mathbf{x}}$ in this system.

On an arbitrary manifold $M^n$ we simply use this as the definition:

**Definition 2.1.** For a curve $\gamma\colon\ (a,b) \to M$, which we write as $\mathbf{x} = \mathbf{x}(t)$, we define the *velocity vector*

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$$

at $t \in (a,b)$ to be the array of numbers

$$(\dot{x}^1, \dots, \dot{x}^n) = \left(\frac{dx^1}{dt}, \dots, \frac{dx^n}{dt}\right)$$

given for each coordinate system $x^1, \dots, x^n$, where $x^i = x^i(t)$, $i = 1, \dots, n$ is the coordinate expression of the curve $\mathbf{x} = \mathbf{x}(t)$. We may also use the notation such as $\dot{\gamma}$ or $d\gamma/dt$ for the velocity of a curve $\gamma$.

Let us emphasize the following. We have not yet defined what a 'vector' on a manifold is in general. Our definition of the velocity vector of a curve on $M$ will serve as a model for such a definition, which will be given below. Hence it is very important to understand Definition 2.1. According to it, the velocity vector for a curve $\mathbf{x} = \mathbf{x}(t)$ at a given $t$ is defined as a rule associating an array (which may be written as a row-vector or row-column) to each coordinate system near $\mathbf{x}(t)$. All these arrays are interpreted as "representing" the velocity in different coordinate systems. (Unlike $\mathbb{R}^n$, there is no distinguished coordinate system and hence there is no distinguished representation of the velocity vector as an element of $\mathbb{R}^n$.)

Let us see how the components of velocity in different coordinate systems are related with each other. Suppose we have coordinates that we denote by $x^1, \dots, x^n$ and another coordinates that we denote by $x^{1'}, \dots, x^{n'}$. (We shall refer to them as 'old' and 'new' coordinates, though these names do not carry any absolute meaning.) Then in the old coordinates the components of velocity are $\dot{x}^i$, where $i = 1, \dots, n$, and in the new coordinates they are $\dot{x}^{i'}$, where $i' = 1', \dots, n'$. By the chain rule we have

$$\dot{x}^i = \sum_{i'} \frac{\partial x^i}{\partial x^{i'}}\, \dot{x}^{i'} \tag{2.3}$$

where the Jacobi matrix is taken at $\mathbf{x}(t)$. (More precisely, if the old coordinates are written as functions of the new coordinates as $x^i = x^i(x^{1'}, \dots, x^{n'})$, the partial derivatives are taken at $x^{1'}, \dots, x^{n'}$ corresponding to the point $\mathbf{x}(t)$.) We see that the transformation law (2.3) depends on a point of $M$.

### 2.1.2 Definition of a tangent vector

We use the transformation law for the components of velocity as the model for a general definition of vectors. Consider a point $\mathbf{x} \in M$ of a manifold $M$.

**Definition 2.2.** A *vector* (or a *tangent vector*) $\mathbf{v}$ at a point $\mathbf{x} \in M$ is defined as a rule assigning an array of numbers $(v^1, \ldots, v^n)$ to each coordinate system $x^1, \ldots, x^n$ near $\mathbf{x}$ so that for any two coordinate systems, say, $x^1, \ldots, x^n$ and $x^{1'}, \ldots, x^{n'}$ the respective arrays are related by the transformation

$$v^i = \sum_{i'} \frac{\partial x^i}{\partial x^{i'}} v^{i'} \tag{2.4}$$

where the partial derivatives are taken at the values of the coordinates $x^{1'}, \ldots, x^{n'}$ corresponding to the point $\mathbf{x}$.

The numbers $v^i$ are called the *components* or *coordinates* of the vector $\mathbf{v}$ w.r.t. a given coordinate system. Below we shall discuss how tangent vectors can be specified practically.

The transformation law for components of vectors defined by (2.4) is called the *vector law*. Note that it depends on a point of $M$, since the Jacobi matrix is not, in general, a constant matrix. Therefore tangent vectors are attached to points. (It makes no sense to speak of a 'vector' on a manifold without referring to a particular point. This is a big difference with $\mathbb{R}^n$.) For example, the velocity vector $\dot{\mathbf{x}}$ of a curve $\mathbf{x} = \mathbf{x}(t)$ is a tangent vector on $M$ at the point $\mathbf{x}(t) \in M$.

The set of all tangent vectors at $\mathbf{x} \in M$ is called the *tangent space* at $\mathbf{x}$ and denoted $T_{\mathbf{x}}M$.

## 2.2 Tangent space

### 2.2.1 Properties

**Theorem 2.1.** *The tangent space $T_{\mathbf{x}}M$ is non-empty. With respect to the evident operations (defined component-wise), it is a vector space of dimension $n = \dim M$, for each $\mathbf{x} \in M$.*

*Proof.* To prove that $T_{\mathbf{x}}M$ is non-empty, one can simply bring up an example of the velocity vector for some curve through $\mathbf{x}$. More precisely, if $\gamma\colon\ t \mapsto \mathbf{x}(t)$ is a curve such that $\mathbf{x}(t_0) = \mathbf{x}$ (for some $t = t_0$), then $\dot{\mathbf{x}}(t_0)$ is an element of

$T_{\mathbf{x}}M$. An alternative way of showing the existence of tangent vectors could be as follows. Fix a coordinate system near $\mathbf{x}$; suppose the coordinates are denoted $x^1, \ldots, x^n$. To define a tangent vector at $\mathbf{x}$, take an arbitrary array $(v^1, \ldots, v^n) \in \mathbb{R}^n$. We want to view $v^i$, $i = 1, \ldots, n$, as the components of a tangent vector $\mathbf{v}$ w.r.t. the coordinate system $x^i$. We need to define the components of $\mathbf{v}$ in all other coordinate systems; we do so by setting

$$v^{i'} = \sum \frac{\partial x^{i'}}{\partial x^i} v^i$$

(i.e., by using the vector law). For consistency, we need to check that the vector law holds for any two arbitrary coordinate systems, say, $x^{i'}$ and $x^{i''}$. We have

$$v^{i''} = \sum \frac{\partial x^{i''}}{\partial x^i} v^i$$

and can express $v^i$ in terms of $v^{i'}$:

$$v^i = \sum \frac{\partial x^i}{\partial x^{i'}} v^{i'} \,.$$

Combining these formulas, we arrive at

$$v^{i''} = \sum \sum \frac{\partial x^{i''}}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} v^{i'} = \sum \frac{\partial x^{i''}}{\partial x^{i'}} v^{i'} \,,$$

(where we used the chain rule).

Now we need to prove that $T_{\mathbf{x}}M$ has the structure of a vector space. To define it, fix again some coordinate system and define the sum of two vectors and the multiplication of a vector by a number componentwise:

$$(\mathbf{u} + \mathbf{v})^i := u^i + v^i, \quad \text{and} \quad (k\mathbf{u})^i := ku^i$$

where $\mathbf{u}, \mathbf{v} \in T_{\mathbf{x}}M$ and $k \in \mathbb{R}$. We have to check that this definition does not depend on a choice of coordinates. Indeed, transform $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ into another coordinate system using the vector law:

$$(\mathbf{u} + \mathbf{v})^{i'} = \sum \frac{\partial x^{i'}}{\partial x^i} (\mathbf{u} + \mathbf{v})^i = \sum \frac{\partial x^{i'}}{\partial x^i} (u^i + v^i) = \\ \sum \frac{\partial x^{i'}}{\partial x^i} u^i + \sum \frac{\partial x^{i'}}{\partial x^i} v^i = u^{i'} + v^{i'} \,,$$

which shows that the expression for the sum of two vectors will have the same form in all coordinate systems. This holds for the multiplication of a vector by a number as well. $\square$

A conclusion from the proof above is that to define a tangent vector it is sufficient to define its components in a particular coordinate system. Such components can be chosen absolutely arbitrarily — no restrictions. Hence, the tangent space $T_{\mathbf{x}}M$ can be identified with $\mathbb{R}^n$. (Such an identification is not unique and depends on a choice of coordinates near $\mathbf{x}$.)

**Remark 2.1.** We used velocities of parametrized curves as a model for defining tangent vectors. In fact, for a given point $\mathbf{x} \in M$, every vector $\mathbf{v} \in T_{\mathbf{x}}M$ is the velocity vector for some curve through $\mathbf{x}$. Indeed, in an arbitrary (but fixed) coordinate system near $\mathbf{x}$ set $x^i(t) = x^i + tv^i$, where $x^i$ are the coordinates of the point $\mathbf{x}$ and $v^i$, the components of the vector $\mathbf{v}$. It is curve passing through $\mathbf{x}$ at $t = 0$ and we have $\dot{\mathbf{x}}(0) = \mathbf{v}$. (The curve looks as a 'straight line' in the chosen coordinate system, but in another chart it would have a different appearance.)

Each coordinate system near $\mathbf{x} \in M$ defines a basis in $T_{\mathbf{x}}M$. The basis vectors are $e_i = \frac{\partial \mathbf{x}}{\partial x^i}$. Recall that partial derivatives are defined as follows: all independent variables except one are fixed and only one is allowed to vary; then the partial derivative is the ordinary derivative w.r.t. this variable. Hence the vectors $e_i = \frac{\partial \mathbf{x}}{\partial x^i}$ are precisely the velocity vectors of the *coordinate lines*, i.e., the curves obtained by fixing all coordinates but one, which is the parameter on the curve. By applying the definition of the velocity vector we see that in the coordinate system $x^1, \dots, x^n$ the basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are represented by the standard basis vectors of $\mathbb{R}^n$:

$$\mathbf{e}_1 = \frac{\partial \mathbf{x}}{\partial x^1} \quad \text{has components} \quad \frac{\partial}{\partial x^1}(x^1, \dots, x^n) = (1, 0, \dots, 0)$$
$$\mathbf{e}_2 = \frac{\partial \mathbf{x}}{\partial x^2} \quad \text{has components} \quad \frac{\partial}{\partial x^2}(x^1, \dots, x^n) = (0, 1, \dots, 0)$$
$$\dots$$
$$\mathbf{e}_n = \frac{\partial \mathbf{x}}{\partial x^n} \quad \text{has components} \quad \frac{\partial}{\partial x^n}(x^1, \dots, x^n) = (0, 0, \dots, 1)$$

Hence $\mathbf{e}_1, \dots, \mathbf{e}_n$ is indeed a basis in $T_{\mathbf{x}}M$ (for all $\mathbf{x}$ in the region where this coordinate system is defined). It is called the *coordinate basis* (for a given coordinate system), or the basis *associated* with it.

Sometimes when it will be convenient we omit the letter $\mathbf{x}$ and use notation $\frac{\partial}{\partial x^k}$ instead $\frac{\partial \mathbf{x}}{\partial x^k}$:

$$\mathbf{e}_1 = \frac{\partial}{\partial x^1}, \quad \mathbf{e}_2 = \frac{\partial}{\partial x^2}, \dots, \mathbf{e}_n = \frac{\partial}{\partial x^n} \tag{2.5}$$

### 2.2.2 Practical description

Consider practical ways of describing tangent vectors.

Firstly, if we are given a chart, the tangent space at a point $\mathbf{x}$ is the linear span of the coordinate basis vectors $\mathbf{e}_i = \frac{\partial}{\partial x^i}$. This is helpful, in particular, if our manifold is defined as a parametrised surface in some $\mathbb{R}^N$.

**Example 2.3.** For $S^2 \subset \mathbb{R}^3$ consider a parametrization by angles $\theta, \varphi$:

$$x = \cos\varphi \sin\theta,\ y = \sin\varphi \sin\theta,\ z = \cos\theta\,.$$

We obtain the vectors $\mathbf{e}_\theta, \mathbf{e}_\varphi$ as elements of $\mathbb{R}^3$:

$$\mathbf{e}_\theta = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} = \cos\varphi\cos\theta\frac{\partial}{\partial x} + \sin\varphi\cos\theta\frac{\partial}{\partial y} - \sin\theta\frac{\partial}{\partial z}\,,$$

$$\mathbf{e}_\varphi = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi}\frac{\partial}{\partial z} = -\sin\varphi\cos\theta\frac{\partial}{\partial x} + \cos\varphi\cos\theta\frac{\partial}{\partial y}$$

$(\frac{\partial}{\partial x} = \mathbf{e}_x,\ \frac{\partial}{\partial y} = \mathbf{e}_y,\ \frac{\partial}{\partial z} = \mathbf{e}_z)$

In other notations

$$\mathbf{e}_\theta = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta)\mathbf{e}_\varphi \quad = (-\sin\varphi\sin\theta, \cos\varphi\sin\theta, 0)\,.$$

The tangent space $T_{\mathbf{x}}S^2$ at $\mathbf{x} \in S^2$ can be described as the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_\theta, \mathbf{e}_\varphi$.

Secondly, if a manifold is specified by equations, then the tangent space at each point appears as the space of solutions of the corresponding "linearized" system.

Consider the simplest case when the manifold is specified by the equation $F(x^1, \ldots, x^N) = 0$ (the condition that vector $\frac{\partial F}{\partial x^i}|_{F=0} \neq 0$ has to be obeyed).

Then to obtain the equation for the tangent space we assume that a vector $\mathbf{v} \in T_{\mathbf{x}}M$ is the velocity of a curve $\mathbf{x}(t)$ on $M$. By differentiating the above equation w.r.t. $t$, we arrive at the equation

$$\sum_{i=1}^{N} \frac{\partial F(x^1, \ldots, x^N)}{\partial x^i}\dot{x}^i = 0\,.$$

Note that the coefficients of the matrix of this system are functions of a point $\mathbf{x} \in M$. This equation specifies a subspace of $\mathbb{R}^N$, the tangent space $T_{\mathbf{x}}M$

at every point of the manifold. The dimension of this tangent space is equal to $N-1$. This means that linear system above have $N-1$ independent solutions, i.e. the vector

$$\left(\frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2}, \ldots, \frac{\partial F}{\partial x^1}\right)$$

is not equal to zero at all the points $F = 0$ of the manifold $M$. This is just the condition that set $F = 0$ is a manifold of dimension $N-1$ (see the Theorem 1.4).

**Example 2.4.** For the sphere $S^2 \subset \mathbb{R}^3$, if we specify it by the equation

$$x^2 + y^2 + z^2 = 1\,,$$

we obtain by differentiating

$$x\dot{x} + y\dot{y} + z\dot{z} = 0\,,$$

(after dividing by 2). We see that $(x, y, z) \neq 0$ on the points where $x^2 + y^2 + z^2 = 1$. For each $\mathbf{x} = (x, y, z) \in S^2$, it is the equation of the tangent space $T_{\mathbf{x}}S^2$ as a subspace of $\mathbb{R}^3$ (of dimension 2) (See also homework 3).

## 2.3 Tangent bundle and tangent maps

### 2.3.1 Tangent bundle

Consider a manifold $M = M^n$.

**Definition 2.3.** The union of the tangent spaces $T_{\mathbf{x}}M$ for all points $\mathbf{x} \in M$, i.e., the collection of all tangent vectors to $M$, is called the *tangent bundle* of $M$ and denoted $TM$,

$$TM = \bigcup_{\mathbf{x} \in M} T_{\mathbf{x}}M\,. \tag{2.6}$$

Let us emphasize that tangent spaces at different points are, by the definition, different vector spaces, which cannot have common elements. Hence the union (2.6) is a disjoint union.

**Remark 2.2.** One should not be confused by the picture of tangent planes to a surface in $\mathbb{R}^3$ where the planes seem to intersect: the 'tangent planes' at this picture are not the actual tangent spaces of the corresponding manifold $M = M^2$, but their images under the maps $i_{\mathbf{x}}\colon\ \mathbf{v} \mapsto \mathbf{x} + \mathbf{v}$ where both $\mathbf{x} \in M$ and $\mathbf{v} \in T_{\mathbf{x}}M$ are regarded as elements of $\mathbb{R}^3$.

There is a natural map $p\colon TM \to M$ that send each vector $\mathbf{v} \in T_{\mathbf{x}}$ to the point $\mathbf{x}$ to which it is attached. It is called the *projection* on $TM$.

The set $TM$ has a natural structure of a manifold of dimension $2n$, induced by the manifold structure of $M^n$.

Consider smooth atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ on $M$. We define the atlas for $TM$ corresponding to this atlas in the following way: Any chart $(U_\alpha, \varphi_\alpha)$ defines a chart $(\bar{U}_\alpha, \bar{\varphi}_\alpha)$ such that

$$\bar{U}_\alpha = \{(\mathbf{x}, \mathbf{v})\colon \ \mathbf{x} \in U_\alpha, \mathbf{v} \in T_{\mathbf{v}}M\}$$

and

$$\bar{\varphi}\colon \ ((\mathbf{x}, \mathbf{v})) \mapsto (x^1, x^2, \dots, x^n; v^1, v^2, \dots, v^n)$$

where $(x^1, x^2, \dots, x^n)$ are coordinates of the point $\mathbf{x}$ in the chart $\varphi_\alpha$, $\varphi(\mathbf{x}) = (x^1, x^2, \dots, x^n)$ and $(v^1, v^2, \dots, v^n)$ are components of the vector $\mathbf{v}$ in these coordinates.

The changes of coordinates consist of the changes of coordinates on $M$ and the corresponding transformations of the components of vectors:

$$x^i = x^i(x^{1'}, \dots, x^{n'})\,, v^i = \sum_{i'} \frac{\partial x^i}{\partial x^{i'}}(x^{1'}, \dots, x^{n'})\, v^{i'}\,,$$

where we denoted by $x^i, v^i$ coordinates in the chart $\tilde{\varphi}_\alpha$ and by $x^{i'}, v^{i'}$ coordinates in the chart $\tilde{\varphi}_\beta$.

The projection map $p\colon TM \to M$ in coordinates has the form

$$(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto (x^1, \dots, x^n)\,,$$

i.e., the standard projection $U_\alpha \times \mathbb{R}^n \to U_\alpha$. We see that the tangent bundle locally looks like a direct product (but it is not a product globally.)

### 2.3.2 Tangent map

Consider a smooth map $F\colon M \to N$. Fix a point $\mathbf{x} \in M$ and denote $\mathbf{y} = F(\mathbf{x}) \in N$. We shall define a linear map from the vector space $T_{\mathbf{x}}M$ to the vector space $T_{\mathbf{y}}N$ 'induced' by the map $F$.

Each tangent vector $\mathbf{v} \in T_{\mathbf{x}}M$ can be interpreted as the velocity vector of some curve $\gamma\colon \ (a, b) \to M$ through $\mathbf{x}$:

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}\,.$$

For concreteness let us assume that $(a, b) = (-\mathbf{e}, \mathbf{e})$ and $\mathbf{x} = \mathbf{x}(0)$; the time derivative above is therefore also taken at $t = 0$.

Consider the composition $F \circ \gamma$; it is a smooth curve in $N$.

**Definition 2.4.** The *tangent map* for the map $F$ at $\mathbf{x} \in M$ maps a vector $\mathbf{v} = \dot{\mathbf{x}} \in T_{\mathbf{x}}M$ to the velocity vector of the curve $F \circ \gamma$ in $N$ at $t = 0$:

$$\dot{\mathbf{x}} \mapsto \frac{d}{dt} F(\mathbf{x}(t)) \tag{2.7}$$

(the derivative is taken at $t = 0$). Notation for the tangent map: $dF(\mathbf{x})$ or $TF(\mathbf{x})$ or $DF(\mathbf{x})$ or $F_*(\mathbf{x})$. (The indication to a point $\mathbf{x}$ is often dropped if it is clear from the context.)

The tangent map for a smooth map $F$ is also referred to as the *differential* of $F$ (at a point $\mathbf{x} \in M$).

**Theorem 2.2.** *The tangent map is a linear transformation $T_{\mathbf{x}}M \to T_{F(\mathbf{x})}N$.*

*Proof.* Suppose $\mathbf{v} \in T_{\mathbf{x}}M$ is the velocity of curve $\gamma\colon (-\mathbf{e}, \mathbf{e}) \to M$ at $t = 0$. Let us calculate the action of $dF(\mathbf{x})$ on $\mathbf{v}$ using coordinates. Fix coordinate systems on $M$ and $N$ so that coordinates on $M$ are denoted as $x^i$, $i = 1 \ldots, n$, and on $N$, as $y^a$, $a = 1 \ldots, m$. Let the curve $\gamma$ be represented in coordinates as $x^i = x^i(t)$, $i = 1, \ldots, n$, and the map $F$, as $y^a = y^a(x^1, \ldots, x^n)$. We have $v^i = \dot{x}^i(0)$. The image of $\mathbf{v}$ under $dF(\mathbf{x})$ is the vector

$$\frac{d}{dt}\big(F(\mathbf{x}(t))\big)\,,$$

(the derivative at $t = 0$), hence in coordinates it will be

$$(dF(\mathbf{x})(\mathbf{v}))^a = \frac{d}{dt}\, y^a\big(x^1(t), \ldots, x^n(t)\big) = \sum_{i=1}^{n} \frac{\partial y^a}{\partial x^i}\frac{dx^i}{dt} = \sum_{i=1}^{n} \frac{\partial y^a}{\partial x^i}\, v^i\,.$$

Hence the tangent map $dF$ at $\mathbf{x} \in M$ is a linear map (as claimed), with the matrix

$$\left(\frac{\partial y^a}{\partial x^i}(x^1, \ldots, x^n)\right)$$

w.r.t. the coordinate bases $\mathbf{e}_i$ in $T_{\mathbf{x}}M$ and $\mathbf{e}_a$ in $T_{F(\mathbf{x})}N$. □

**Example 2.5.** Consider a smooth function on $M$, i.e., a map $f\colon\ M \to \mathbb{R}$. We identify tangent vectors to $\mathbb{R}$ with numbers. Hence the tangent map $df(\mathbf{x})$ at a point $\mathbf{x} \in M$ is a linear map $T_{\mathbf{x}}M \to \mathbb{R}$. In coordinates it is given by the matrix (row-vector)

$$\left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right) .$$

The value of $df(\mathbf{x})$ on a tangent vector $\mathbf{v}$ with coordinates $v^i$ will be

$$df(\mathbf{x})(\mathbf{v}) = \frac{\partial f}{\partial x^1} v^1 + \ldots + \frac{\partial f}{\partial x^n} v^n .$$

Thus we have recovered the classical understanding of the differential of a function as the main (linear) part of the increment of the function for a given increment of the independent variables. The value of $df(\mathbf{x})$ on $\mathbf{v}$ is traditionally denoted as $\partial_{\mathbf{v}} f$ and is called the *derivative along* $\mathbf{v}$ (sometimes also "directional derivative", but it is not a good term, as it depends on a vector $\mathbf{v}$ and not just on its direction).

**Example 2.6.** Consider the map $F$ of $S^1$ in $\mathbb{R}^3$, which is defined by the relation

$$x = \cos\theta,\ y = \sin\theta,\ z = \sin\left(\theta + \frac{\pi}{4}\right), \tag{2.8}$$

where $\theta$ is the polar angle on the circle $S^1$ defined by the equation $x^2 + y^2 = 1$ in $\mathbb{R}^2$. These relations define the smooth map, because functions $\cos\theta, \sin\theta$ are smooth periodical functions: the transformations $\theta \mapsto \theta - 2\pi$ does not change these functions.

To be more pedantic the coordinate $\theta\colon 0 < \theta < 2\pi$ is defined in all the points except the point $(1, 0)$.

One can consider another coordinate: the angle $\theta'\colon\ -\pi < \theta < \pi$ which is defined at this point. The transition functions between coordinate $\theta, \theta'$ are $\theta' = \theta$ or $\theta' = \theta - 2\pi$. Hence the relations (2.9) are well-defined at all $\theta$ [3]

[3] It is useful to consider the following **counterexample**, the map

$$x = \cos\theta,\ y = \sin\theta,\ z = c\theta, \tag{2.9}$$

This map defines the helix for $0 < theta < 2\pi$, but this does not define the smooth map from $S^1$ to $\mathbb{R}^3$.

Take any point $\mathbf{x}_0 = \theta_0$ on the circle and tangent vector $\dot{\mathbf{x}} = v\frac{\partial}{\partial\theta}$ at this point. Then $dF_{\mathbf{x}_0}(\dot{\mathbf{x}})$ is the vector in $\mathbb{R}^3$:

$$dF_{\mathbf{x}_0}(\dot{\mathbf{x}}) = \left(\frac{\partial x}{\partial\theta}v, \frac{\partial y}{\partial\theta}v, \frac{\partial z}{\partial\theta}v\right) = \left(-\sin\theta_0 v, v\cos\theta_0, v\cos\left(\theta_0 + \frac{\pi}{4}\right)\right)$$

Suppose we have two smooth maps, $G\colon M \to N$ and $F\colon N \to P$. Consider their composition $F \circ G\colon M \to P$. Let $\mathbf{x} \in M$, $\mathbf{y} = G(\mathbf{x}) \in N$, and $\mathbf{z} = F(\mathbf{y}) \in P$. How to calculate the tangent map to the composition $F \circ G$ at $\mathbf{x}$? It is supposed to be a linear transformation $T_{\mathbf{x}}M \to T_{\mathbf{z}}P$.

**Theorem 2.3.** *The tangent map $d(F \circ G)$ for the composition $F \circ G$ at $\mathbf{x}$ is the composition of the linear transformations $dF$ at $\mathbf{y} = F(\mathbf{x})$ and $dG$ at $\mathbf{x}$:*

$$d(F \circ G)(\mathbf{x}) = dF(\mathbf{y}) \circ dG(\mathbf{x})\,. \tag{2.10}$$

*Proof.* One way of proving this is to write down both sides in coordinates. If $x^1, \ldots, x^n$ are coordinates on $M$, $y^1, \ldots, y^m$ are coordinates on $N$, and $z^1, \ldots, z^r$ are coordinates on $P$, and the maps $F$ and $G$ are specified, respectively, by the equations $z^\mu = z^\mu(y^1, \ldots, y^m)$ and $y^a = y^a(x^1, \ldots, x^n)$, then the LHS of (2.10) has the matrix

$$\left(\frac{\partial z^\mu}{\partial x^i}\right) \tag{2.11}$$

and the matrix of the RHS is the product of matrices

$$\left(\frac{\partial z^\mu}{\partial y^a}\right)\left(\frac{\partial y^a}{\partial x^i}\right) = \left(\sum_a \frac{\partial z^\mu}{\partial y^a}\frac{\partial y^a}{\partial x^i}\right), \tag{2.12}$$

which coincides with (2.11) by the chain rule known from multivariate calculus. In other words, the composition formula (2.10) is but an abstract version of the chain rule. It is possible to come to (2.10) more geometrically, working directly from the definition. Suppose $\mathbf{v} \in T_{\mathbf{x}}M$ is the velocity of a curve $\mathbf{x}(t)$ at $t = 0$. Then the image of $\mathbf{v}$ under $d(F \circ G)(\mathbf{x})$ is

$$\frac{d}{dt}F(G(\mathbf{x}(t))) \tag{2.13}$$

(at $t = 0$). On the other hand, the image of $dG(\mathbf{x})(\mathbf{v})$ under $dF(\mathbf{y})$, i.e., the RHS of (2.10) applied to $\mathbf{v}$, is

$$\frac{d}{dt}F(\mathbf{y}(t)) \tag{2.14}$$

for an arbitrary curve $\mathbf{y}(t)$ on $N$ such that its velocity at $t = 0$ is $dG(\mathbf{x})(\mathbf{v})$. We can take $\mathbf{y}(t) := G(\mathbf{x}(t))$ as such a curve, and plugging it into (2.14), we arrive at (2.13). Hence the LHS and the RHS of (2.10) coincide. $\square$

Since there are maps $dF(\mathbf{x})\colon T_{\mathbf{x}}M \to T_{F(\mathbf{x})}N$ for all points $\mathbf{x} \in M$, they assemble to a map of tangent bundles, denoted $dF$ or $TF$ or $DF$ or $F_*$:

$$dF\colon TM \to TN\,.$$

Properties of the tangent map $dF$ can be used for studying properties of a smooth map $F$.

A map $F$ is called a *submersion* or we say that $F$ is *submersive* if the linear map $dF(x)$ is an epimorphism (i.e., surjective) at each point $x$. It is called an *immersion* (or an *immersive* map) if the linear map $dF(x)$ is a monomorphism (i.e., injective) at each point.

One can show, by using the implicit function theorem, that every immersion is "locally" an injective map. However, it is possible for a smooth map to be injective but not immersive. (Example: the map $\mathbb{R} \to \mathbb{R}, x \mapsto x^3$.) A smooth map $F\colon M \to N$ that is both injective and immersive has a very important property: its image $F(M) \subset N$ has a natural structure of a smooth manifold and, moreover, it is what is called a *submanifold* of $N$. We skip the details referring the reader to textbooks on differentiable manifolds. When we say that there is an *embedding* of one manifold into the other we understand that there is a map that is both injective and immersive.

# 3 Topology of a manifold

Last updated: January 15, 2009.

## 3.1 Topology induced by manifold structure

Recall what is the topology.

Let $X$ be a set covered by the family $\mathcal{F} = \{U_\alpha\}$ of subsets such that the following conditions are obeyed:

- empty set $\varnothing$ and the whole set $X$ belongs to the family $\mathcal{F}$, , $\varnothing \in \mathcal{F},\ X \in \mathcal{F}$ $\quad *$

- For any subset $I$ of indices the union $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{F}$. $\quad **$

- 

  For any finite subset of indices $I$, $(|I| < \infty)$ $\bigcap_{\alpha\in I} U_\alpha \in \mathcal{F}$. $\quad * * *$

  In other words the union of any subfamily of $\mathcal{F}$ and intersection of any finite subfamily of sets in $\mathcal{F}$ belong to $\mathcal{F}$ [4]

Then the family $\mathcal{F}$ is called *topology* on $X$ and the space $X$ equipped with the family $\mathcal{F}$ is called *topological space*. The subsets $U_\alpha$ are called *open* sets.

**Example 3.1.** The basic example of topological space is $\mathbb{R}^n$ equipped with the family $\mathcal{F} =$ open sets where "open" means to be open in standard sense (the set is open if any point is internal)

Now we define a topology on a manifold compatible with the structure of smooth manifold.

Consider a differentiable (smooth) manifold $M = (M, \mathcal{A})$, where the atlas $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$, and $\varphi_\alpha \colon U_\alpha \to V_\alpha)$ are charts on the open domains in $\mathbb{R}^n$.

**Definition 3.1.** A subset $A \subset M$ is *open* if and only if for each chart $(U_\alpha, \varphi_\alpha)$, the set $\varphi_a(A \cap U_\alpha) \subset \mathbb{R}^n$ is.

**Example 3.2.** Each $U_\alpha$ is an open set. Indeed by definition of the atlas for an arbitrary chart $\varphi_\beta$, $\varphi_\beta(U - \alpha \cap U_\beta)$ are open in $\mathbb{R}^n$. Hence each $U_\alpha$ is open.

**Example 3.3.** $M$ itself is an open. Indeed for an arbitrary chart $\varphi_\alpha$, $\varphi_\alpha(M \cap U_\alpha) = \varphi_\alpha(U_\alpha) = V_\alpha$ is open in $\mathbb{R}^n$. Hence $M$ is open.

**Example 3.4. Counterexample** Take an arbitrary point $\mathbf{x} \in M$. Show that this is not an open set. Take a chart $(U_\alpha, \varphi_\alpha)$ such that $\mathbf{x} \in U_\alpha$. We see that $\varphi_\alpha(\mathbf{x})$ is a point in $\mathbb{R}^n$. It is not open set. Hence $\mathbf{x}$ is not an open set in manifold $M$.

One can easy to check that

**Theorem 3.1.** *Open subsets of a manifold $M$ (defined as above) satisfy the axioms of a topology.* $\square$

[4] Note that empty set can be considered as a union of empty subfamily of sets in $\mathcal{F}$ and the whole set can be considered as a intersection of empty family of sets in $\mathcal{F}$. Hence the first axiom follows from the second and third one.

E.g. if we consider intersection $A \cap B$ of two open sets $A, B$. Then for an arbitrary chart $(U_\alpha, \varphi_a)$

$$\varphi_\alpha \left( (A \cap B) \bigcap U_\alpha \right) = \varphi_\alpha \left( (A \cap U_\alpha) \bigcap (B \cap U_\alpha) \right) \varphi_\alpha \left( (A \cap U_\alpha) \right) \bigcap \varphi_\alpha \left( (B \cap U_\alpha) \right)$$

is the open set in $\mathbb{R}^n$, because it is the intersection of two open sets in $\mathbb{R}^n$. Hence $A \cap B$ is open set in $M$.

Therefore each manifold can be considered as a topological space. We shall refer to the topology defined above as to the *manifold topology* or the topology given by a manifold structure.

What is the relation between smoothness and continuity?

**Theorem 3.2.** *A smooth map is continuous.*

**Remark 3.1.** The statement of this theorem seems to be foolish, because

we often told that smooth function is continuous and it has infinitely many derivatives. But these statements were made about smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. We never did before any statements about continuity of functions defined on manifold, because topology was not defined.

Recall such fundamental notions as compactness and connectedness (and path-connectedness).

1. The topological space $X$ is called compact if an arbitrary covering with open sets contains finite subcovering, i.e. if for the family $\{V_a\}$ of open sets ($V_\alpha \in \mathcal{F}$)

$$X = \bigcup V_\alpha \,, \tag{3.1}$$

then there exist a finite subcollection $\{V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_k}\}$ such that

$$X = V_{\alpha_1} \cup V_{\alpha_2} \cup V_{\alpha_3} \cup \cdots \cup V_{\alpha_{k-1}} \cup V_{\alpha_k} \tag{3.2}$$

2. The topological space $X$ is called connected if for an arbitrary two open sets $A, B \in \mathcal{F}$ the following condition is satisfied

$$X = A \cup B \& A \cap B = \varnothing \Rightarrow A = \varnothing \,\mathrm{or}\, B = \varnothing \tag{3.3}$$

or in other words the topological space is not connected if there exist two non-empty open sets $A, B$ such that their intersection is empty but their union covers the space $X$.

E.g. $S^n$ is compact and connected.

## 3.2 Tangent vectors as derivations

After we have introduced topological notions, we can come back to analysis on manifolds.

Let $M = M^n$ be a manifold. Consider a vector $\mathbf{v} \in T_{\mathbf{x}}M$. Suppose $\gamma\colon (-\mathbf{e}, \mathbf{e}) \to M$ is a curve such that $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ at $t = 0$.

**Definition 3.2.** For an arbitrary function $f\colon M \to \mathbb{R}$ the number

$$\partial_{\mathbf{v}} f := \frac{d}{dt} f(\mathbf{x}(t))\,.$$

(the derivative at $t = 0$) is called the *derivative of $f$ along* $\mathbf{v}$.

Comparing it with the definition of the differential, we see that

$$\partial_{\mathbf{v}} f = df(\mathbf{v})\big|_{\mathbf{x}}\,.$$

If $x^1, \ldots, x^n$ are coordinates near $\mathbf{x}$ and $v^i$, $i = 1, \ldots, n$ are the components of $\mathbf{v}$, we have

$$\partial_{\mathbf{v}} f = \sum_{i=1}^{n} v^i \, \frac{\partial f}{\partial x^i}(\mathbf{x}).$$

**Proposition 3.1.** *The operation $\partial_{\mathbf{v}}\colon\ C^\infty(M) \to \mathbb{R}$ satisfies the following properties: linearity over $\mathbb{R}$ and the Leibniz rule:*

$$\partial_{\mathbf{v}}(\lambda f + \mu g) = \lambda \partial_{\mathbf{v}} f + \mu \partial_{\mathbf{v}} g\,, \forall \lambda, \mu \in R \tag{3.4}$$

$$\partial_{\mathbf{v}}(fg) = \partial_{\mathbf{v}} f \cdot g(\mathbf{x}) + f(\mathbf{x}) \cdot \partial_{\mathbf{v}} g\,. \tag{3.5}$$

*Proof.* Immediate. ∎

Note that the map sending a function $f \in C^\infty(M)$ to the number $f(\mathbf{x}) \in \mathbb{R}$ is a homomorphism, called the *evaluation homomorphism at* $\mathbf{x}$. Sometimes denoted $\mathrm{ev}_{\mathbf{x}}$.

There is a fundamental notion.

**Definition 3.3.** Suppose $\alpha\colon A_1 \to A_2$ is a homomorphism of algebras. A linear map $D\colon A_1 \to A_2$ is called an *derivation* over the homomorphism $\alpha$ if there is a 'Leibniz rule'

$$D(ab) = D(a) \cdot \alpha(b) + \alpha(a) \cdot D(b)$$

for all $a, b \in A_1$.

The derivative $\partial_{\mathbf{v}}$ in the Proposition above was just a derivation of algebra $C^\infty(M)$ over homomorphism $\mathrm{ev}_{\mathbf{x}}\colon f \mapsto f(\mathbf{x})$

It turns out that **all** the derivations of the algebra of functions on a manifold to numbers are of the form $\partial_{\mathbf{v}}$. To show this, let us explore first the case of $\mathbb{R}^n$.

**Theorem 3.3.** *Let $\mathbf{x}_0 \in \mathbb{R}^n$. For an arbitrary derivation $D\colon\ C^\infty(\mathbb{R}^n) \to \mathbb{R}$ over the evaluation homomorphism $\mathrm{ev}_{\mathbf{x}_0}$ there exists a vector $\mathbf{v} \in T_{\mathbf{x}_0}\mathbb{R}^n$ such that $D = \partial_{\mathbf{v}}$.*

In other words this theorem states that if $D$ is linear operation on functions with values in real numbers such that at the point $\mathbf{x}_0$ the Leibnitz rule is satisfied:

$$D(fg) = Dfg(\mathbf{x}_0) + f(\mathbf{x}_0)Dg\,,$$

then there exists a vector $\mathbf{v}$ tangent at the point $\mathbf{x}_0$ to $\mathbb{R}^n$ such that

$$D = \partial_{\mathbf{v}}$$

This theorem justifies our notation of vector fields by derivatives.

The proof uses the following simple but fundamental statement:

**Lemma 3.1 (Hadamard's Lemma).** *For any smooth function $f \in C^\infty(\mathbb{R}^n)$ and any $\mathbf{x}_0 \in \mathbb{R}^n$ there is an expansion*

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n}(x^i - x_0^i)g_i(\mathbf{x}) \tag{3.6}$$

*where $g_i \in C^\infty(\mathbb{R}^n)$ are some smooth functions.*

*Proof.* Consider the segment joining $\mathbf{x}$ and $\mathbf{x}_0$ and write

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \int_0^1 \frac{d}{dt} f(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0))\,dt =$$

$$f(\mathbf{x}_0) + \sum (x^i - x_0^i) \int_0^1 \frac{\partial f}{\partial x^i}(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0))\,dt$$

□

**Corollary 3.1.**

$$g_i(\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}_0} = \frac{\partial f}{\partial x^i}\big|_{\mathbf{x}=\mathbf{x}_0} \tag{3.7}$$

*Proof.* Differentiating left and right hand sides of the equation (3.6) (all functions are smooth) we come to the relation:

$$\frac{\partial f \mathbf{x}}{\partial x^i} = g_i(x) + \sum (x^k - x_0^k)\frac{\partial g(\mathbf{x})}{\partial x^i}$$

Put $\mathbf{x} = \mathbf{x}_0$ in this relation and we come to equation (3.7) $\square$

Now we can prove the main theorem.

*Proof of Theorem 3.3.* Let $D\colon\ C^\infty(\mathbb{R}^n) \to \mathbb{R}$. Apply $D$ to the expansion (3.6). First note that derivations kill constants; indeed, $D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1)$, hence $D(1) = 0$ and then $D(c) = D(c \cdot 1) = 0$ for any $c \in \mathbb{R}$. Therefore using (3.6) and (3.7) we obtain that

$$D(f) = D(f(\mathbf{x}_0) + \sum_i (x^i - x_0^i) g_i) = \sum D(x^i) g_i(\mathbf{x}_0) = \sum_i v^i \frac{\partial f}{x^i}|_{\mathbf{x}=\mathbf{x}_0} \quad (3.8)$$

for $\mathbf{v} = (v^1, \ldots, v^n)$ where $v^i = D(x^i)$. $\square$

We see that

Combining the statement of Theorem 3.3 and Proposition 3.1 we see that the derivations of the algebra $C^\infty(\mathbb{R}^n) \to \mathbb{R}$ over the homomorphism $\mathrm{ev}_{\mathbf{x}_0}$ are in one-to-one correspondence with the tangent vectors $\mathbf{v} \in T_{\mathbf{x}} M$.

This can be generalised to an arbitrary differentiable manifold.

**Theorem 3.4.** *For an arbitrary point $\mathbf{x}_0$ on the differentiable manifold $M$ the derivations of the algebra $C^\infty(M) \to \mathbb{R}$ over the evaluation homomorphism $\mathrm{ev}_{\mathbf{x}_0}$ are in one-to-one correspondence with the tangent vectors $\mathbf{v} \in T_{\mathbf{x}_0} M$.*

*In other words for an arbitrary point $\mathbf{x}_0$ an arbitrary vector $\mathbf{v} \in T_{\mathbf{x}_0} M$ defines the linear operation $\partial_{\mathbf{v}}$ which satisfies at the point $\mathbf{x}_0$ the Leibnitz rule and vice versa if $D$ is a linear operation satisfying the Leibnits rule at the point $\mathbf{x}_0$ then $D = \partial_{\mathbf{v}}$.*

This is a reason why we identify vectors $\mathbf{v}$ with the corresponding derivations $\partial_{\mathbf{v}}$. For example, the coordinate basis vectors $\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial x^i}$ are identified with the partial derivatives $\partial_i = \frac{\partial}{\partial x^i}$.

We do not prove the Theorem.

## 3.3 Bump functions and partitions of unity

We have to admit an embarrassing fact: for a general manifold $M$ we do not have tools allowing to show the existence of smooth functions defined everywhere on $M$ (besides constants). On $\mathbb{R}^n$ we have plenty of functions: first of all, the coordinate functions $x^i$, then polynomials and various other smooth functions of $x^1, \dots, x^n$. By contrast, in the absence of global coordinates, how one can find a non-trivial smooth functions on a manifold?

How we are sure that there exists at least one smooth function which is not equal to zero at all the points? [5]

It turns out that it is necessary to impose **topological restrictions** on a manifold $M$ to guarantee a good supply of $C^\infty$ functions. It is done below. (Without them, one can construct "pathological" examples where the only smooth functions are constants. On the other hand, with the restrictions described below, we shall be able to show that a manifold can be embedded into a Euclidean space of sufficiently large dimension, which guarantees an abundance of smooth functions.)

We impose on manifolds the following topological conditions

- it is Haussdorff topological space, i.e. for an arbitrary two different points $a \neq b$ one can find two open sets $U_a, U_b$ such that $a \in U_a, b \in U_b$ and $U_a \cap U_b = \varnothing$

- The atlas is countable, i.e. it contains countable number of charts [6]. Later on we will consider in fact only the cases when manifold is compact and atlas is finite

**Remark 3.2.** Here and later speaking about topology of manifold we mean the *topology induced by manifold structure* (see the beginning of this section) In particularly the set is *open* if its image under an arbitrary chart is an open domain in $\mathbb{R}^n$ (see the beginning of this section).

[5] Note that in the complex analysis situation is very rigid: if a function is equal to zero in the vicinity of an internal point of an analiticity domain, then it is equal identically to zero in the whole domain

[6] this condition areises from the axiom of second countability: The topological space is called second countable if it has a countable base. (A base of a topological space is a family of open sets such that an arbitrary open set is the union of sets from that family)

**Remark 3.3.** Note that an arbitrary smooth atlas on compact manifold can be reduced to the finite atlas. Moreover in all non"pathological" cases manifold is Haussdorffian and tone can deal with finite smooth atlas.

With the conditions imposed above it is possible to show that there are enough smooth functions.

Recall that the *support* of a function (notation: $\mathrm{Supp}\, f$) is the closure of the subset where the function does not vanish.

**Lemma 3.2 (Bump functions).** *For any point $\mathbf{x}_0 \in M$ of a smooth manifold $M$ there is a nonnegative smooth function $g_{\mathbf{x}_0} \in C^\infty(M)$ (a "bump function") globally defined on the whole $M$ and compactly supported in a neighborhood of $\mathbf{x}_0$ and which is identically $1$ on a smaller neighborhood.*

Formulate a more simplified version of this lemma:

**Lemma 3.3 (Bump functions (simplified)).** *For any point $\mathbf{x}_0 \in \mathbb{R}^n$ and an arbitrary open ball $B_\varepsilon(\mathbf{x}_0) = \{\mathbf{x}\colon d(\mathbf{x}, \mathbf{x}_0) < \varepsilon\}$ there is a nonnegative smooth function globally defined on the whole $M$ $g_{\mathbf{x}_0} \in C^\infty(\mathbb{R}^n)$ (a "bump function") such that this function is equal to zero out of the ball $B_\varepsilon(\mathbf{x}_0)$ and it is identically $1$ on a smaller ball $B_{\frac{\varepsilon}{2}}(\mathbf{x}_0)$.*

**Remark 3.4.** In the case if open domain is a ball $B_\varepsilon(\mathbf{x}_0)$ then its closure is the closed ball $\bar{B}_\varepsilon(\mathbf{x}_0) = \{\mathbf{x}\colon d(\mathbf{x}, \mathbf{x}_0) \le \varepsilon\}$. The fact that this function is equal to zero out of the open ball $B_\varepsilon(\mathbf{x}_0)$ means that $\mathrm{Supp}\, f$ (the support of the function) belongs to the open ball $B_\varepsilon(\mathbf{x}_0)$.

The fact that smooth function $f$ is equal 1 in the open ball $B_{\frac{\varepsilon}{2}}(\mathbf{x}_0)$ means that it is equal to 1 in the closure $\bar{B}_{\frac{\varepsilon}{2}}(\mathbf{x}_0)$ of this ball.

**Remark 3.5.** Note that the non-trivial part of this statement is that this function is defined on the whole $M$. E.g. if we consider complex analytical functions on $\mathbb{C}$ then in this class of functions we *cannot* define bump function: the condition that complex analytical function is equal identically to constant in the vicinity of any point leads to the fact that it is equal to the constant in the whole complex plane.

*Proof.* First let us prove the following. Let $C_a$ be a cube $\{|x^i| \le a, \forall i\}$ in $\mathbb{R}^n$. Prove that there is a $C^\infty$-function which equals zero outside $C_2$ and equals to one on $C_1$. Consider the function

$$\Phi(x) = \begin{cases} e^{-\frac{1}{x}} \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases} \tag{3.9}$$

$f(x)$ defined as $e^{-1/x}$ for $x > 0$ and zero for $x \leq 0$. It is well-known fact that this function is smooth (see for e.g. solutions of coursework). Now consider the function $H(x)$ on
$R$ such that

$$H(x) = \frac{\Phi(x)}{\Phi(x) + \Phi(1-x)} .$$

One can see that this is well-defined smooth function and

$$H(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x \geq 1 \end{cases} \tag{3.10}$$

It is easy to see that the function

$$G(x) = H(x+2)H(2-x) \tag{3.11}$$

is equal to zero out of the open interval $(-2, 2)$ and it is equal to 1 in the closed interval $[-1, 1]$. Taking the product of such functions for each coordinate, we obtain the required function on $\mathbb{R}^n$.

To transfer this proof to a manifold (i.e. to prove non-simplified version of the lemma), one has to use the Hausdorff condition; and we skip the details. The details are as follows. Consider a manifold $M$ and a point $\mathbf{x} \in M$. Consider a coordinate neighborhood $V$ around $\mathbf{x}$. From the local construction we have a smooth function $g_{\mathbf{x}}$ defined on $V$ such that it is 1 near $\mathbf{x}$ and its support is contained in some open $O \subset V$ homeomorphic to an open ball or a cube. Extend $g_{\mathbf{x}}$ from $V$ to $M \setminus V$ by zero. It is necessary to prove that the result is smooth. It is sufficient to prove that for any $\mathbf{y} \in M \setminus V$ there is a whole neighborhood $W$ on which $g_{\mathbf{x}}$ is identically zero. Fix $\mathbf{y}$. Consider all points $\mathbf{z} \in \bar{O}$. Since $M$ is Hausdorff, there are disjoint open neighborhoods $O_{yz}$ and $O_{yz}$ of $\mathbf{z}$ and $\mathbf{y}$ respectively. Since $\bar{O}$ is compact, we can extract a finite number of neighborhoods $O_{yz_k}$, $k = 1, \ldots, N$ covering $\bar{O}$. Then the intersection $\bigcap O_{yz_k}$ is an open neighborhood of $\mathbf{y}$ and does not intersect with $\bar{O}$. So we can take it as $W \ni y$, and the function $g_{\mathbf{x}}$ is identically zero on $W$, hence smooth at $\mathbf{y}$.

□

Now we study *partition of unity*

Let $M$ be a smooth manifold and $\{U_\alpha\}$, $\alpha = 1, \ldots, N$ is covering of this manifold by open sets: $M = \bigcup_{\alpha=1}^N U_\alpha$. We say in this case that the manifold $M$ is covered by the finite collection $\{U_\alpha\}$ of open sets.

**Definition 3.4.** Let a manifold $M$ is covered by the finite collection $\{U_\alpha\}$ of open sets ($\alpha = 1, 2, \ldots, N$).

We say that a collection of smooth non-negative functions $\{\rho_\alpha\}$, (globally defined on the whole $M$) where $\alpha = 1, 2, \ldots, N$ is *a smooth partition of unity subordinate to the cover* $\{U_\alpha\}$ if the following conditions are obeyed

- Support of every function $\rho_\alpha$, i.e. the closure of the points where $\rho_\alpha \neq 0$ belongs to the open set $U_\alpha$:

$$\mathrm{Supp}\,\rho_\alpha \subset U_\alpha$$

- all functions $\rho_\alpha$ are non-negative functions

$$\rho_\alpha \geq 0$$

- sum of these functions is equal to 1 (at any point $x \in M$):

$$\sum_{\alpha=1}^{N} \rho_\alpha(x) = 1. \tag{3.12}$$

**Theorem 3.5.** *For any finite cover $\{U_\alpha\}$ of the manifold $M$ there exists a partition of unity subordinate to this open cover.*

**Remark 3.6.** In particularly if $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$ is a finite smooth atlas of manifold $M$, $\alpha = 1, \ldots, N$, where $N$ is number of charts, then the manifold $M$ is covered by the collection $\{U_\alpha, \varphi_\alpha\}$ of open sets and one can consider a partition of unity subordinated to this cover.

The proof uses the bump functions. We omit the proof, and consider the following very simple examples

**Example 3.5.** Consider the following two open sets on $\mathbb{R}$: $U_+ = (-\varepsilon, \infty)$ and $U_- = (-\infty, \varepsilon)$, where $\varepsilon > 0$. These two open sets cover $\mathbb{R}$: $\mathbb{R} = U_= \cup U_+$. Construct smooth partition of unity subordinate to this covering. Consider non-negative functions

$$G_+(x) = H(x), \qquad G_-(x) = H(\delta - x), \text{ where } 0 < \delta < \varepsilon$$

where $H(x)$ is the bump function (3.10).

For all points $x > 0$ the function $G_+(x) > 0$ and for all $x < \delta$ $G_-(x) < 0$. Hence one can consider the following two smooth functions well-defined on the whole $\mathbb{R}^n$:

$$\rho_+(x) = \frac{G_+(x)}{G_+(x) + G_-(x)}, \quad \rho_-(x) = \frac{G_-(x)}{G_+(x) + G_-(x)} \tag{3.13}$$

It is evident that

$$\rho_\pm \geq 0, \mathrm{Supp}\,\rho_+ \subset U_+,\ \mathrm{Supp}\,\rho_- \subset U_-,\ \text{and}\ \rho_+(x) + \rho_-(x) \equiv 1,$$

i.e. $\{\rho+, \rho_-\}$ is the smooth partition of unity subordinate to the cover $\{U_+, U_-\}$ of the $\mathbb{R}$.

**Example 3.6.** A partition of unity for the open cover of $S^2$ consisting of $U_N = S^2 \setminus \{N\}$ and $U_S = S^2 \setminus \{S\}$. (see solutions of coursework(after 18november))

Having bump functions is already sufficient for showing that there are many global smooth functions. Indeed, take an arbitrary smooth function defined in a neighborhood of $\mathbf{x} \in M$. Then, by multiplying it by a suitable bump function, it is possible to extend it by zero (from a smaller neighborhood) to the whole manifold $M$. We shall use this many times in the future.

Now we formulate very important and very beautiul statement—-Urysohn Lemma.

**Corollary 3.2.** *(***Urysohn lemma)** *Let $C_1, C_2 \subset M$ be closed subsets such that $C_0 \cap C_1 = \varnothing$. There is a function $f \in C^\infty(M)$ such that $f \equiv 1$ on $C_1$ and $f \equiv 0$ on $C_0$.*

(Recall the subset $A$ in the topological space $X$ is closed if and only if $X \backslash A$ is open.)

*Proof.* Consider the open cover $M = U_1 \cup U_2$, where $U_1 := M \setminus C_1$ and $U_2 := M \setminus C_2$. The intersection of sets $C_1, C_2$ is empty, hence the union of the sets $U_1, U_2$ cover $M$. Consider a smooth partition of unity $\{\rho_1, \rho_2\}$ subordinate to this cover. $\rho_1 + \rho_2 = 1$. Take $f := \mathbf{r}_1$. By definition, $\mathrm{Supp}\, f_1 \subset U_1 = M \setminus C_1$. Thus $f = 0$ on $C_1$. Similarly, $\rho_2 = 0$ on $C_2$. Thus $f = \rho_1 = 1 - \rho_2 = 1$ on $C_1$. □

**Remark 3.7.** Note that the function $f$ which we constructed satisfies already the additional condition

$$0 \leq f \leq 1 \quad \text{for all points of the manifold } M \tag{3.14}$$

Functions constructed in Corollary 3.2 are called *Urysohn functions* for the pair $C_0, C_1$. On a topological space the existence of (continuous) Urysohn functions is a separation property which is stronger than being Hausdorff.

## 3.4 Embedding manifolds in $\mathbb{R}^N$

Smooth manifold can be embedded in $\mathbb{R}^N$. It is one of the remarkable theorems of mathematics.

"Manifolds" first appeared as surfaces in $\mathbb{R}^N$. They were later axiomatised to give abstract manifolds we consider. However this theorem shows that the abstract viewpoint yields the same supply of objects.

**Theorem 3.6.** *Smooth compact manifold can be embedded in $\mathbb{R}^N$ for sufficiently large $N$.*

To prove the theorem we assume that manifold can be covered by the so called atlas with "thick boundaries"

Let $M = M^n$ be a compact smooth manifold, and $\mathcal{A} = \{U_a, \varphi_a\}$, $(\alpha = 1, \dots, N)$ be a finite atlas on it.

We say that an atlas of $N$ charts is an atlas with *thick boundaries* if the following conditions are obeyed:

- All $\varphi_a$ are maps onto open unit balls in $\mathbb{R}^n$, i.e. all local coordinates $x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n$ run over the unit balls in $\mathbb{R}^n$:

$$(x_\alpha^1)^2 + (x_\alpha^2)^2 + (x_\alpha^3)^2 + \dots + (x_\alpha^n)^2 < 1$$

- The open sets $\{W_\alpha\}$ defined as

$$W_\alpha = \{\mathbf{x}\colon\ \mathbf{x} \in U_\alpha \ \text{ such that } \ (x_\alpha^1)^2 + (x_\alpha^2)^2 + (x_\alpha^3)^2 + \dots + (x_\alpha^n)^2 < s^2\}$$

  also cover the manifold $M$.

Any open set $U_\alpha$ is the preimage of the ball of the radius $r = 1$ under the map $\varphi_\alpha^{-1}$ and an open set $W_\alpha$ can be considered as the preimage of a ball of radius $r = s < 1$ under the map $\varphi_\alpha^{-1}$.

Usually one takes $s = \frac{1}{2}$.

In other words we say that atlas with thick boundaries is provided if manifold is covered by unit balls with thick boundaries such that interior of these balls having radius $s$ also cover manifold $M$.

Suppose that the given atlas is the atlas with thick boundaries.

Proof of the Theorem.

Now consider the set of "bump" functions $\{\sigma_\alpha\}$ chosen such that

$$\sigma(\mathbf{x})\colon \begin{cases} \sigma_\alpha(\mathbf{x}) = 1 \text{ if } \mathbf{x} \in \overline{W}_\alpha, \text{ i.e. if } (x_\alpha^1)^2 + (x_\alpha^2)^2 + (x_\alpha^3)^2 + \cdots + (x_\alpha^n)^2 \leq \frac{1}{4} \\ 0 < \sigma_\alpha(\mathbf{x}) < 1 \text{ if } \mathbf{x} \in U_\alpha, \text{ but } \mathbf{x} \notin \overline{W}_\alpha, \text{ i.e. if} \frac{1}{4} < (x_\alpha^1)^2 + \cdots + (x_\alpha^n)^2 < 1 \\ \sigma_\alpha(\mathbf{x}) \equiv 0 \text{ if } \mathbf{x} \notin U_\alpha, \text{ i. if } \ (x_\alpha^1)^2 + \cdots + (x_\alpha^n)^2 \geq 1 \end{cases} \tag{3.15}$$

(Note that these functions define the partition of unity subordinate to this atlas: $\rho_\alpha(\mathbf{x}) = \frac{\sigma_\alpha(\mathbf{x})}{\sum \sigma_\beta(\mathbf{x}})$)

Consider the $N(n+1)$-dimensional coordinate space $\mathbb{R}^{N(n+1)}$. Coordinates of this space we denote by $y_\alpha^i$, where $i = 1, 2, 3, \ldots, n, n+1$, and $\alpha = 1, 2, 3, \ldots, N$. (You may view $y_\alpha^i$ as $N \times n+1$ matrices.). Our goal is to embed our smooth manifold $M^n$ in this $N(n+1)$-dimensional linear space $\mathbb{R}^{N(n+1)}$.

We define the following map $\Phi\colon M^n \to \mathbb{R}^{N(n+1)}$. Let $\mathbf{x}$ be an arbitrary point on manifold $M$, let $(U_\alpha, \varphi_\alpha)$ be an arbitrary chart of the atlas. Then we define the value $\Phi(\mathbf{x})$ in $\mathbb{R}^{N(n+1)}$ by the following formula:

$$\begin{cases} y_\alpha^i(\mathbf{x}) = \sigma_\alpha(\mathbf{x}) x_a^i, (i = 1, \ldots, n) \text{ and } y_\alpha^{n+1}(\mathbf{x}) = \sigma_\alpha(\mathbf{x}) \text{ if } \mathbf{x} \in U_\alpha \\ y_\alpha^i(\mathbf{x}) = 0, (i = 1, \ldots, n+1) \ \text{ if } \mathbf{x} \notin U_\alpha \end{cases} \tag{3.16}$$

where functions $\{\sigma_\alpha(\mathbf{x})\}$ are defined in (3.15).

This map is smooth map from manifold $M^n$ to $\mathbb{R}^{N(n+1)}$. Let us show that it is an embedding.

Recall that a map $\Phi\colon M \to N$ is an embedding if it is an injection and induces an injection of tangent spaces, i.e. if the following conditions are obeyed:

- $\Phi(\mathbf{x}_1) = \Phi(\mathbf{x}_2) \Leftrightarrow \mathbf{x}_1 = \mathbf{x}_2$
- It is monomorphism of tangent spaces for an arbitrary point $\mathbf{x} \in M$: For two tangent vectors $\mathbf{v}_1, \mathbf{v}_2$ at a point $\mathbf{x}$

$$d\Phi_{\mathbf{x}}(\mathbf{v}_1) = d\Phi_{\mathbf{x}}(\mathbf{v}_2) \Leftrightarrow \mathbf{v}_1 = d\Phi\mathbf{v}_2\,.$$

We show first that our $\Phi$ is a monomorphism of tangent spaces then that it is injection. We need to calculate differential in an arbitrary point $\mathbf{x}$. The matrix of differential is the rectangular matrix of the order $n \times (N(n+1))$. We need to show that this matrix has the maximal rank, i.e. its rank is equal to $n$. Take $\alpha$ such that $\mathbf{x} \in W_\alpha$. In the vicinity of this point $\sigma_\alpha(\mathbf{x}) \equiv 1$, hence

$$\begin{cases} y^i_\alpha(\mathbf{x}) = x^i_\alpha, 1 \leq i \leq n \\ y^{n+1}_\alpha(\mathbf{x}) = 1 \end{cases}$$

We see that derivatives matrix $\frac{\partial y^i_\alpha}{\partial x^j_\alpha}$ is identity matrix. Hence "big" matrix has maximal rank $n$.

It remains to prove the injection of manifolds.

Take an arbitrary $\mathbf{x}_1, \mathbf{x}_2$. Suppose they belong to the same $W_\alpha$ and $\mathbf{x}_1 \neq \mathbf{x}_2$. Then $\sigma_\alpha(\mathbf{x}_1) = \sigma_\alpha(\mathbf{x}_2) = 1$ and they have different coordinates and and $y^i_\alpha(\mathbf{x}_1) \neq y^i_\alpha(\mathbf{x}_2)$.

Suppose now that $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to different balls, $\mathbf{x}_1 \in W_{\alpha_1}$, $\mathbf{x}_2 \in W_{\alpha_2}$. Then $y^{n+1}_{\alpha_1}(\mathbf{x}_1) = 1$ but $y^{n+1}_{\alpha_1}(\mathbf{x}_2) \neq 1$.

We see that this map is injection. Thus we proved that the map (3.16) is injection and it is immersion.

The Theorem is proved.

One can prove that the dimension of the ambient space can be decreased till $N = 2n+1$. This is the content of Withney Theorem

# 4 Vector fieldls and their commutators

Consider a manifolds $M$ and tangent bundle $TM$. Consider the map $\pi$— projection. It is a surjection of $TM$ on $M$. The map $s\colon M \to TM$ such that $\pi \circ s = \mathrm{id}$ is an identity is a section, *or a vector field*.

The vector field assigns to every point $\mathbf{x} \in M$ the vector $s(\mathbf{x})$ belonging to the tangent space $T_\mathbf{x}M$.

A vector field $\mathbf{X}$ on a manifold $M$ is the smooth map from $M$ to $TM$.

We often consider vector fields defined not on the whole manifolds but on its open domains.

## 4.1 Commutator of vector fields

The space of vector fields is linear space. One can add vector fields, multiply them on constants.

Now we define a fundamental operation: commutator of vector fields.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two arbitrary vector fields defined in domain $U$ of manifold.

**Definition 4.1.** Vector field $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ is called commutator of vector fields $\mathbf{X}, \mathbf{Y}$ if for an arbitrary function $f$ the following identity holds:

$$\partial_{\mathbf{C}} f \equiv \partial_{\mathbf{X}} \partial_{\mathbf{Y}} f - \partial_{\mathbf{Y}} \partial_{\mathbf{X}} f$$

One can see that this definition is correct. (See exercises in homework 4)

# 5 Differential forms

Consider manifold $M$ and its tangent bundle $TM$.

Consider a manifold $M = M^n$ and the tangent space $T_{\mathbf{x}}M$ at a point $\mathbf{x} \in M$. An exterior $k$-form on $T_{\mathbf{x}}M$, i.e., an antisymmetric tensor at $\mathbf{x}$ of order $k$ is called a *differential $k$-form* at $\mathbf{x}$.

Using local coordinates near $\mathbf{x}$, we can write a differential $k$-form at $\mathbf{x}$ as

$$\omega = \omega_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k} (k = 0, 1, 2, 3, \ldots, n)$$

where the coefficients are numbers. If we allow these numbers to vary, we get a *differential $k$-form* on an open domain $U \subset M$. The adjective 'differential' is often dropped. In the case if $k \geq n$ exterior $k$-form vanishes because antisymmetrical tensor with many than $n$ components vanishes.

## 5.1 Exterior differential

The space of all $k$-forms on a manifold $M$ is denoted $\Omega^k(M)$ and likewise for any open $U \subset M$.

A particular case of a differential form is, of course, a function. Functions are the same as 0-forms. Recall that we have **differential** on functions, which takes functions to covector fields, i.e., in the new terminology, to 1-forms:

$$d\colon \;\; C^\infty(M) = \Omega^0(M) \to \Omega^1(M)\,.$$

It turns out that it is possible to extend this operation to obtain an operation that maps $k$-forms to $(k+1)$-forms for all $k \geq 0$.

**Theorem 5.1.** *(**Theorem-Definition**) There exists a unique map $d\colon\ \Omega^k(U) \to \Omega^{k+1}(U)$ for all open sets $U \subset M$ and each $k = 0, 1, \ldots$ , with the following properties:*

1. *$d$ is linear over $\mathbb{R}$: $d(\lambda\omega + \mu\omega') = \lambda d\omega + \mu d\omega'$ for $\lambda, \mu \in \mathbb{R}$.*

2. *$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma$ if $\omega \in \Omega^k(U)$;*

3. *for all $f \in \Omega^0(U) = C^\infty(U)$, $df$ is the usual differential of functions;*

4. *for all $f \in C^\infty(U)$, $d(df) = 0$;*

5. *$d$ commutes with restrictions, i.e., if there are two open sets $V \subset U$ and $\omega \in \Omega^k(U)$, then $(d\omega)|V = d(\omega|V)$.*

*The map $d$ is called the* ***exterior differential*** *or* ***exterior derivative****.*

*The following relation holds*

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U) \xrightarrow{d} 0$$

*where $d^2 = 0$.*

*Proof.* Suppose a map possessing the above properties exists. We shall show that it is unique and obtain for it an explicit formula. Then we shall use this formula as a definition and check that it indeed satisfies the required properties, and thus prove existence. Consider a domain $U \subset M$ where it is possible to introduce local coordinates. Let $d$ exist on $U$ and satisfy the properties 1–4. Let $\omega = \omega_{i_1 \ldots i_k}(\mathbf{x})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ be a coordinate representation of a $k$-form $\sigma \in \Omega^k(U)$. Then, by using 1 and 2, we arrive at

$$d\omega = d\omega_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} + \omega_{i_1 \ldots i_k} d\left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) .$$

By 3, $d\omega_{i_1 \ldots i_k}$ is the usual differential of a function. Applying 2, 3, and 4, we deduce that $d\left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) = 0$. Indeed, $d(dx^i) = 0$ for all $i$; we have

$$d\left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) = d(dx^{i_1}) \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} - dx^{i_1} \wedge d\left(dx^{i_2} \wedge \ldots \wedge dx^{i_k}\right) = 0 \,,$$

by induction. Therefore, if $d$ exists, it should be given by the formula

$$d\omega = d\omega_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \,. \tag{5.1}$$

In particular, it follows that this expression should hold in any coordinate system on $U$. Conversely, let us now define $d$ on $U$ by formula (5.1) using some particular coordinate system, and check that it satisfies the axioms 1–4. Properties 1 (linearity) and 3 are clearly satisfied. To check 2, it is sufficient to consider forms $\omega$ and $\sigma$ of the form $\omega = f dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and

$\sigma = g dx^{j_1} \wedge \ldots \wedge dx^{j_l}$, where $f$ and $g$ are arbitrary functions. The general case follows by linearity. We have

$$\begin{aligned} d(\omega\sigma) = d\left(f dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge g dx^{j_1} \wedge \ldots \wedge dx^{j_l}\right) = \\ d\left(fg\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}\right) = \\ d(fg) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} = \\ \left(g df + f dg\right) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} = \\ df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge g dx^{j_1} \wedge \ldots \wedge dx^{j_l} + (-1)^k f dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dg \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}\,, \end{aligned}$$

where the sign $(-1)^k$ comes from swapping $dg$ and $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, and it is precisely $d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma$. Hence 2 is proved. Consider now 4. Let $f$ be a function. We have

$$df = \frac{\partial f}{\partial x^i}\, dx^i.$$

Hence

$$d(df) = d\left(\frac{\partial f}{\partial x^i}\, dx^i\right) = \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i = -\frac{\partial^2 f}{\partial x^i \partial x^j}\, dx^i \wedge dx^j = -d(df) = 0.$$

Here we used the commutativity of partial derivatives and the skew-commutativity of the wedge product. Hence 4 is proved. To summarize, we proved the existence and uniqueness of the operator $d$ for any domain $U \subset M$ admitting local coordinates (from properties 1–4). If $V \subset U$ is an open subset of such a domain $U$, then we can use the restriction of coordinates from $U$ to $V$ to calculate $d$ on $V$. Hence 5 will also be satisfied. For an arbitrary open domain (such as $M$ itself) we use a coordinate cover. To define $d$ on 'global' forms, we consider their restrictions to coordinate domains and apply $d$ there; uniqueness of $d$, which is established for coordinate domains, implies that coordinate expressions for $d$ agree on intersections. Hence $d$ is defined globally. Properties 1–5 follow. □

**Remark 5.1.** Antisymmetry plays crucial role in the extension of the operation $d$ from functions to forms of higher degree. It is impossible to construct an analog of the exterior differential for other types of tensor fields.

Consider examples of calculation of the exterior derivative.

**Example 5.1.** Let $A$ be a 1-form given in local coordinates by $A = A_j\, dx^j$. We have $dA = dA_j \wedge dx^j$. By expanding $dA_j$ we arrive at $dA = \partial_i A_j\, dx^i \wedge dx^j$.

This is not the final expression yet, because the coefficients $\partial_i A_j$ are not antisymmetric (note that the summation is over all combinations of indices $i, j$). By writing $\partial_i A_j = \frac{1}{2}(\partial_i A_j - \partial_j A_i) + \frac{1}{2}(\partial_i A_j + \partial_j A_i)$ we finally obtain

$$dA = \frac{1}{2}\,(\partial_i A_j - \partial_j A_i)\, dx^i \wedge dx^j = \sum_{i<j} (\partial_i A_j - \partial_j A_i)\, dx^i \wedge dx^j\,, \tag{5.2}$$

which is the desired formula for the differential of 1-forms.

## 5.2 Integral of a form over $\mathbb{R}^n$

Let us first recall integration of a function over $\mathbb{R}^n$. Fix a coordinate system on $\mathbb{R}^n$. Denote the corresponding coordinates $x^1, \ldots, x^n$. (For example, we can take the standard coordinates.)

Consider a function $f \in C^\infty(\mathbb{R}^n)$ and suppose that it has compact support. Practically that means that $f$ is identically zero outside of a sufficiently large cube. Consider the integral of $f$ over $\mathbb{R}^n$ **with respect to a chosen coordinate system**. The integral makes sense because $f$ is compactly supported. It is, effectively, the integral over a sufficiently large cube:

$$\int_{\mathbb{R}^n} f(x) dx^1 \ldots dx^n := \int_{C_R} f(x) dx^1 \ldots dx^n\,.$$

Here $x = (x^1, \ldots, x^n)$ and $C_R = \{|x^i| \leq R \text{ for } \forall i = 1, \ldots, n\}$.

What happens if we change coordinates: $x = x(x')$, or, in greater detail, $x^i = x^i(x^{1'}, \ldots, x^{n'})$. It is well known that the following formula holds:

$$\int_{\mathbb{R}^n} f(x) dx^1 \ldots dx^n = \int_{\mathbb{R}^n} f(x(x'))\, |J(x')|\, dx^{1'} \ldots, dx^{n'} \tag{5.3}$$

where

$$J(x') = \det\Big(\frac{\partial x^i}{\partial x^{i'}}\Big)$$

is the Jacobian of the given transformation of coordinates, i.e., the determinant of the Jacobi matrix. Note that the formula contains the absolute value $|J|$ of the Jacobian, not just the Jacobian $J$.

**Remark 5.2.** One might be confused about this absolute value. In particular, aren't there a discrepancy with the formula for functions of a single variable? In it,

$$\int_a^b f(x) dx = \int_{a'}^{b'} f(x(x'))\, \frac{dx}{dx'}\, dx'\,,$$

there arises the derivative $\frac{dx}{dx'}$ itself, not the absolute value. The explanation is as follows. The notation $\int_a^b$ for the integral over a segment includes a choice of the order of the endpoints $a, b$: that $a$ comes first, $b$, second. In fact, it is the integral over an oriented segment (see below) and it includes an extra sign if $a > b$. Therefore the formula for a change of variable takes care of this extra sign. Suppose $a < b$ so that the LHS can be understood as the integral over the segment in the same sense as we write $\int_{\mathbb{R}^n}$ above. If $\frac{dx}{dx'} > 0$, then $a' < b'$ too, and both sides can be interpreted in the same sense. Here $\frac{dx}{dx'} = |\frac{dx}{dx'}|$. If $\frac{dx}{dx'} < 0$, then $a' > b'$ and the integral in the RHS is the negative of the integral over the segment "without orientation". This extra minus combines with $\frac{dx}{dx'}$ under the integral sign to give the factor of $-\frac{dx}{dx'} = |\frac{dx}{dx'}|$, exactly as in the general formula (5.3).

Fix an orientation on $\mathbb{R}^n$. Suppose an $n$-form $\omega \in \varnothing^n(\mathbb{R}^n)$ has compact support (meaning that the coefficient $\omega_{12\ldots n}(x)$ has compact support).

**Definition 5.1.** The *integral* of $\omega$ over $\mathbb{R}^n$ with a chosen orientation is defined as

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \omega_{12\ldots n}(x)\, dx^1 \wedge \ldots \wedge dx^n := \int_{\mathbb{R}^n} \omega_{12\ldots n}(x)\, dx^1 \ldots dx^n$$

where at the RHS stands the integral w.r.t. an arbitrary coordinate system (belonging to the given orientation) of the function $\omega_{12\ldots n}$, the component of $\omega$ in this system.

It immediately follows that the integral of a differential form does not depend on a choice of coordinates:

$$\int_{\mathbb{R}^n} \omega_{12\ldots n}(x)\, dx^1 \wedge \ldots \wedge dx^n = \int_{\mathbb{R}^n} \omega_{1'2'\ldots n'}(x')\, dx^{1'} \wedge \ldots \wedge dx^{n'}\,,$$

provided the orientation is preserved.

Instead of integrating forms over the whole $\mathbb{R}^n$ one can consider integrals over (oriented) open domains $U \subset \mathbb{R}^n$. In this case it should be assumed that an $n$-form to be integrated is compactly supported inside $U$, i.e., the closure of the set where the form does not vanish is contained in $U$ (and is compact).

**Remark 5.3.** Recall two bases $(\mathbf{e}_i)$ and $(\mathbf{e}_{i'})$ define the same orientation if the transition matrix between them has positive determinant. The above definition for coordinate systems can be rendered as follows: two coordinate systems define the same orientation if an only if the corresponding bases $(\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial x^i})$ and $(\mathbf{e}_{i'} = \frac{\partial \mathbf{x}}{\partial x^{i'}})$ define the same orientation as each point. (For connected domains such as the whole $\mathbb{R}^n$ it is sufficient to check at one point only.)

In actual examples it is also necessary to integrate forms over compact domains $D \subset \mathbb{R}^n$ such as, e.g., a closed ball or closed cube.

## 5.3 Integration of forms over manifolds

A manifold is *orientable* if it possesses an atlas for which the Jacobians of all changes of coordinates are positive. Such an atlas is called *orienting*. Two orienting atlases are said to be *equivalent*, or compatible, or belonging to the same orientation, or defining the same orientation, if their union is also orienting. (In other words, the changes of coordinates between charts of the two atlases also have positive Jacobians.) Formally, an *orientation* of a manifold $M$ can be defined as the equivalence class of an orienting atlas. If $M$ is not orientable (= "is non-orientable"), then it does not have an orientation.

It is not difficult to see that if an orientable manifold $M$ is connected, it has exactly two orientations.

An *oriented manifold* is an (orientable) manifold together with some chosen orientation.

Consider a compact oriented manifold $M = M^n$. Choose a finite atlas $(U_\alpha, \varphi_\alpha \colon U_\alpha \to V_\alpha \subset \mathbb{R}^n)$ belonging to the chosen orientation and a partition of unity $(\rho_\alpha)$ subordinate to this atlas, i.e., $\operatorname{Supp} \rho_\alpha \subset U_\alpha$. We may assume that functions $\rho_\alpha$ have compact support (indeed, their supports are closed subsets of a compact space, therefore compact).

**Definition 5.2.** Let $\omega$ be $n$-form. The *integral* of $\omega$ over $M^n$ with a given orientation is defined as follows:

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega\,,$$

where each integral in the sum is defined as

$$\int_{U_\alpha} \rho_\alpha \omega := \int_{V_\alpha} (\varphi_a^{-1})^*(\rho_\alpha \omega)$$

We may rewrite the integral over $V_\alpha$ as the integral over the whole $\mathbb{R}^n$:

$$\int_{U_\alpha} \rho_\alpha \omega = \int_{V_\alpha} (\varphi_a^{-1})^*(\rho_\alpha \omega) = \int_{\mathbb{R}^n} (\varphi_a^{-1})^*(\rho_\alpha \omega)\,,$$

which makes sense because the integrand is compactly-supported. Such integrals do not depend on the choice of coordinates as we have established above.

**Theorem 5.2.** *Definition of the integral of an n-form over a compact oriented manifold $M^n$ does not depend on a choice of atlas and partition of unity.*

*Proof.* First of all we notice that, for a fixed atlas $(U_\alpha, \varphi_\alpha)$, the integral does not depend on a choice of coordinates, i.e., on the the maps $\varphi_\alpha$, as long as they belong to the given orientation. Consider now two atlases $(U_\alpha, \varphi_\alpha)$ and $(U'_\mu, \varphi'_\mu)$ together with partitions of unity $(\rho_\alpha)$ and $(\rho'_\mu)$ subordinate to them. Consider the cover $(W_{\alpha\mu} = U_\alpha \cap U'_\mu)$ and notice that the products $\rho_\alpha \rho'_\mu$ make a partition of unity subordinate to $(W_{\alpha\mu})$. We may write

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \sum_\mu \rho'_\mu \rho_\alpha \omega = \sum_{\alpha,\mu} \int_{U_\alpha} \rho_\alpha \rho'_\mu \omega = \sum_{\alpha,\mu} \int_{U_\alpha \cap U'_\mu} \rho_\alpha \rho'_\mu \omega\,,$$

and similarly

$$\sum_\mu \int_{U'_\mu} f'_\mu \omega = \sum_{\alpha,\mu} \int_{U_\alpha \cap U'_\mu} \rho_\alpha \rho'_\mu \omega\,,$$

which is the same. Hence the two sums are equal and the integral $\int_M \omega$ therefore does not depend on a choice of cover and a partition of unity subordinate to it. ☐

**Remark 5.4.** Definition of integral using partitions of unity has theoretical significance (allows to prove theorems). Practically when calculating integrals one works differently. A rather standard situation is as follows: a manifold $M$ admits an atlas such that one particular chart covers the whole of $M$ except for a set of lower dimension (such as a line for a two-dimensional manifold, etc.). One can check that in such a case the integral over $M$ amounts to the integral over this particular chart, and no partition of unity is necessary.

**Example 5.2.** For $S^1$ we can use angular coordinate $\theta \in (0, 2\pi)$. Or stereographic coordinate. (In both cases a single point is missing.)

**Example 5.3.** Calculate integral over the unit circle with center at the origin in $\mathbb{R}^2$ of the following form:

$$A = \frac{xdy - ydx}{x^2 + y^2} \, .$$

Solution: consider embedding $x = \cos t$, $y = \sin t$, where $t \in [0, 2\pi]$. We have $i^* A = d\theta$, therefore

$$\int_{(S^1, i)} A = \int_0^{2\pi} dt = 2\pi \, .$$

**Example 5.4.** Consider stereographic coordinates on $S^2$. Given a 2-form

$$\omega = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} \, .$$

Calculate its integral over $S^2$. First of all, notice that $S^2$ is orientable and we may assume that an orientation is given by $x, y$. We have

$$\int_{S^2} \omega = \int_{S^2} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} = \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} \, .$$

Substituting $x = r \cos\varphi$, $y = r \sin\varphi$, where $r \geq 0$ and $0 < \varphi < 2\pi$, we arrive at

$$\int_{S^2} \omega = \int_0^{\infty} \int_0^{2\pi} \frac{rdrd\varphi}{(1 + r^2)^2} = \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{1}{2} \frac{d(1 + r^2)}{(1 + r^2)^2} = - \left. \frac{2\pi}{2(1 + r^2)} \right|_{r=0}^{r=\infty} = \pi.$$

## 5.4 Stokes theorem

There is a natural extension of the notion of a manifold.

**Example 5.5.** Closed disk.

Consider $\mathbb{R}^n_+ = \{x^n \geq 0\}$. Open sets can be of two types.

Change of coordinates between two open domains involving boundary points.

**Lemma 5.1.** *At the points of the boundary, $x^n = 0$,*

$$\frac{\partial x^n}{\partial x^{n'}} > 0 \, .$$

**Definition 5.3.** A *manifold with boundary* is defined exactly as an ordinary manifold (manifolds are sometimes referred to as 'manifolds without boundary') with the replacement of open sets in $\mathbb{R}^n$ by open sets in $\mathbb{R}^n_+$. The subset consisting of all points where the last coordinate is zero is called the *boundary* and denoted $\partial M$.

We immediately see that $\partial M$ is itself a manifold (without boundary).
From the above Lemma we have

**Corollary 5.1.** *Orientation of $M$ induces orientation on $\partial M$.*

Convention: positive orientation on $\partial M$ is $(-1)^{n-1}\times$ the orientation given by $x^1, \dots, x^{n-1}$ if $x^1, \dots, x^n$ give orientation on $M$.

**Remark 5.5.** If function $f \in C^\infty(\mathbb{R})$ has compact support, then

$$\int_{\mathbb{R}} \frac{df}{dx}\, dx = 0.$$

**Theorem 5.3 (Stokes Theorem).** *For compact oriented manifold with boundary $M^n$ and an $(n-1)$-form $\sigma \in \Omega^{n-1}(M)$,*

$$\int_M d\sigma = \int_{\partial M} \sigma$$

# 6 De Rham cohomology

## 6.1 Definition, examples, and basic properties

A $k$-form $\omega$ is *closed* if $d\omega = 0$. It is *exact* if there is a $(k-1)$-form $\sigma$ such that $d\sigma = \omega$.

The condition $d\omega = 0$ is local, in the sense that a form on $M$ is closed if it is closed near every point of $M$ (in other words, if all the restrictions $\omega|U$ are closed as forms on $U \subset M$). To check that $d\omega = 0$, we simply check it in coordinates.

Compared to it, the property of being exact is not a local one.

We know that every exact form is closed, since $d(d\omega) = 0$. On the other hand, we have already seen examples of closed but not exact forms. As a prototype of all such examples one should consider the 1-form $\omega = d\theta$ on $\mathbb{R}^2 \setminus \{0\}$ where $\theta$ is the polar angle. In standard Cartesian coordinates,

$$\omega = \frac{xdy - ydx}{x^2 + y^2}\,.$$

This form is not exact, because for a cycle going around the origin once, say, counterclockwise,

$$\oint_C \omega = 2\pi\,.$$

(The exact number is not important; that it is non-zero, is important.) On the other hand, $d\theta$ looks like an exact form (the differential of the function $\theta$), but it is not, because the function $\theta$ is 'bad'. It cannot be made everywhere defined, smooth and single-valued function in the whole of $\mathbb{R}^2 \setminus \{0\}$, at the same time. It can be made such an 'honest' function only locally, in a sufficiently small domain, not allowing cycles going around the origin. This is a typical situation. Every closed form can be regarded as exact if we surrender some of the properties: allow discontinuities, multi-valuedness, or consider it only locally. Therefore, the difference between closed forms and exact forms is a 'global' feature and measures the 'topological complexity' of the manifold in question.

**Definition 6.1.** Closed $k$-forms $\omega$ and $\omega'$ on a manifold $M$ are *cohomologous* if their difference is exact: $\omega - \omega' = d\tau$ for some $(k-1)$-form $\tau$. (The form $\tau$ should be well-defined smooth form on the whole $M$.) Equivalence classes w.r.t. this relation are called *cohomology classes.* The set of all cohomology classes of degree $k$ (sometimes people also say: in dimension $k$), is denoted $H^k(M)$.

(Check that it is indeed an equivalence relation!)

Consider the sum of two closed forms, $\omega + \sigma$. It is closed, by the linearity of $d$. If we replace one of the summands by a cohomologous form, the sum will remain in the same cohomology class: $\omega+\sigma' = \omega+(\sigma+d\tau) = (\omega+\sigma)+d\tau$. Thus, $H^k(M)$ inherits the structure of an Abelian group. Similarly, multiplication by real numbers is well-defined on cohomology classes. Therefore $H^k(M)$ is a vector space. Nevertheless, the traditional name for $H^k(M)$ is the $k$-th *cohomology group.*

**Remark 6.1.** It is also called the **de Rham** cohomology group, after George de Rham, who proved a fundamental statement (the "de Rham theorem") about these groups. The name suggests that there are other cohomology theories, and this is indeed true, but for manifolds they all give the same objects. This is the statement of the de Rham theorem. Sometimes the notation $H^k_{DR}(M)$ is used for de Rham cohomology.

Consider the product of closed forms: $\omega \wedge \sigma$. The Leibniz rule implies that it is closed: $d(\omega\wedge\sigma) = d\omega\wedge\sigma \pm \omega\wedge d\sigma = 0+0 = 0$. If one of the factors is replaced by a different representative of the same cohomology class, then the class of the product will not change: $\omega' \wedge \sigma = (\omega + d\tau)\wedge\sigma = \omega\wedge\sigma + d\tau\wedge\sigma = \omega\wedge\sigma + d(\tau\wedge\sigma)$, by the same Leibniz rule, since $d\sigma = 0$. Hence, the exterior multiplication induces a well-defined multiplication on cohomology classes. We arrive at the graded algebra $H^*(M)$, called the *cohomology algebra* of $M$.

Notation for cohomology classes: $[\omega] \in H^k(M)$, if $\omega \in \Omega^k(M)$ is a closed form representing an element $[\omega]$. Exact forms and only them represent the zero class.

**Proposition 6.1.** *For any manifold $M$, the dimension of the space $H^0_{DR}(M)$ is the number of connected components of $M$.*

*Proof.* The space of 0-forms is the space of smooth functions $C^\infty(M)$. A function $f \in \Omega^0(M) = C^\infty(M)$ is closed if $df = 0$. What does it mean? Near each point $f$ must be a constant (indeed, we may introduce coordinates and write $df = 0$ in coordinates). Hence $f$ is a *local constant*. It need not be a constant on the whole $M$ (a global constant), which is demonstrated by the example of a manifold consisting of two connected components such as two disjoint copies of $\mathbb{R}^n$. The function may be zero on one component and 1, on another. Notice that there no exact 0-forms (because there are no $-1$-forms). Therefore $H^0(M)$ is the space of all local constants on $M$. It is intuitively clear and can be proved by a simple topological argument that on a connected topological space any local constant is a constant. Therefore, if a topological space, in particular, a manifold $M$, is the disjoint union of its connected components (maximal connected subspaces), then any local constant is defined by its values of the components (constants); thus it is a function on the set of components. If the number of components is finite, the space of functions on this set is finite-dimensional. As a basis one take functions that are identically 1 on one component and identically 0 on all other components. Hence the dimension is the number of components. $\square$

In most of our examples, the manifolds in question are connected. Then, automatically, $H^0(M) \cong \mathbb{R}$ for them.

Consider examples of cohomology classes in degrees higher than zero.

**Example 6.1.** We saw above that $[d\theta]$ is a non-zero class in $H^1(\mathbb{R}^2 \setminus \{0\})$.

**Example 6.2.** If we consider $\theta$ as a parameter on the circle $S^1$ (defined up to $2\pi$), the 1-form $d\theta$ is well-defined and gives a non-zero class $[d\theta]$ in $H^1(S^1)$. We shall shortly see that all other cohomology classes of the circle in degree 1 are proportional to $[d\theta]$, i.e., $\dim H^1(S^1) = 1$ and $[d\theta]$ can be taken as a basis (consisting of a single element).

**Definition 6.2.** The dimension $\dim H^k(M)$ is denoted $b_k(M)$ and called, the $k$-th *Betti number* of $M$.

Consider a smooth map $F\colon M \to N$. It induces the pull-back of forms: $F^*\colon \Omega^k(N) \to \Omega^k(M)$.

**Proposition 6.2.** *The pull-back on forms induces a linear map of cohomology (denoted by the same symbol):*

$$F^*\colon H^k(N) \to H^k(M)\,.$$

*Moreover, $F^*$ is an algebra homomorphism of cohomology algebras.*

*Proof.* Define $F^*[\omega]$ as $[F^*\omega]$. We have to check that the class $[F^*\omega]$ will not change if we replace $\omega$ by a cohomologous form $\omega + d\sigma$. Indeed,

$$F^*(\omega + d\sigma) = F^*\omega + F^*d\sigma = F^*\omega + d(F^*\sigma)\,.$$

(It is here where we need that pull-backs commute with $d$!) To see that $F^*$ is an algebra homomorphism on cohomology, consider

$$F^*\big([\omega][\sigma]\big) = F^*\big([\omega \wedge \sigma]\big) = [F^*\big(\omega \wedge \sigma\big)] = [F^*\omega \wedge F^*\sigma] = \\ [F^*\omega][F^*\sigma] = F^*[\omega]\, F^*[\sigma]\,.$$

□

**Corollary 6.1.** *If two manifolds are diffeomorphic, then the corresponding de Rham cohomology groups (vector spaces, actually) are isomorphic.*

*Proof.* Suppose $F\colon M \to N$ and $G\colon N \to M$ are mutually inverse smooth maps giving a diffeomorphism $M \cong N$. Consider $H^k(M)$ and $H^k(N)$ for some given $k$. There are linear maps

$$F^*\colon H^k(N) \to H^k(M)$$

and

$$G^*\colon H^k(M) \to H^k(N)\,.$$

The identities $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$ imply $(F \circ G)^* = G^* \circ F^* = \mathrm{id}$ and $(G \circ F)^* = F^* \circ G^* = \mathrm{id}$. Hence $F^*$ and $G^*$ are mutually inverse linear maps, so the vector spaces $H^k(M)$ and $H^k(N)$ are isomorphic (in particular, have the same dimension). $\square$

**Remark 6.2.** Moreover, if $M$ and $N$ are diffeomorphic, the cohomology algebras $H^*(M)$ and $H^*(N)$ are isomorphic. Indeed, we have isomorphism of vector spaces (in each degree) that also preserves the multiplication.

Corollary 6.1 means that the de Rham cohomology is a 'diffeomorphism invariant' of a smooth manifold.

We shall see below that de Rham cohomology is, in fact, an invariant under a much coarser equivalence relation. Namely, it is a 'homotopy invariant'.

## 6.2 Poincaré Lemma and the homotopy property

**Example 6.3.** On the real line $\mathbb{R}$, any 1-form $\omega = f(x)dx$ (which is automatically closed), is exact. Indeed, consider the function

$$F(x) = \int_0^x f(y)dy\,.$$

It is defined on the whole line $\mathbb{R}$ and we have $dF = \omega$ by the Newton–Leibniz theorem (the "main theorem of calculus").

This is a prototype of the general statement concerning $\mathbb{R}^n$ or the so-called 'star-shaped' domains.

**Definition 6.3.** An open domain $U \subset \mathbb{R}^n$ is called *star-shaped* if there is a point $\mathbf{x}_0 \in U$ such that for each $\mathbf{x} \in U$, all points of the segment $[\mathbf{x}_0\mathbf{x}]$ are in $U$. (The segment $[\mathbf{x}_0\mathbf{x}]$ consists of all points of the form $(1-t)\mathbf{x}_0 + t\mathbf{x}$ where $t \in [0,1]$.) The point $\mathbf{x}_0$ is often referred to as the *center* of the star-shaped domain $U$.

For example, any convex domain (i.e., such that for any two points the segment joining them is also contained in the domain) is star-shaped. However, a star-shaped domain does not have to be convex.

**Example 6.4.** An open ball in $\mathbb{R}^n$ is convex, therefore star-shaped.

**Example 6.5.** The interior of any "star" in $\mathbb{R}^2$ is star-shaped. (Notice that a star is not convex.)

**Theorem 6.1 (Poincaré Lemma).** *On $\mathbb{R}^n$ or any star-shaped domain in $\mathbb{R}^n$, every closed $k$-form is exact, if $k \geq 1$.*

*Proof.* We shall only consider the case $k = 1$. (The general case can be considered similarly. Alternatively, it could be deduced from the homotopy invariance considered below.) Let $A = A_i(\mathbf{x})dx^i$ be a closed 1-form on a star-shaped domain $U \subset \mathbb{R}^n$, $dA = 0$. That means that $\partial_i A_j - \partial_j A_i = 0$. Without loss of generality we may assume that the center $\mathbf{x}_0$ of $U$ is the origin $0 \in \mathbb{R}^n$. For any point $\mathbf{x} \in U$, the segment $[0\mathbf{x}]$ consists of points $t\mathbf{x}$ where $t \in [0,1]$. Define a function $f$ by the formula

$$f(\mathbf{x}) = \int_0^1 x^i A_i(t\mathbf{x})dt$$

for all $\mathbf{x} \in U$. It makes sense because $t\mathbf{x} \in U$. We have

$$\begin{aligned}
df &= dx^j \frac{\partial}{\partial x^j}\left(\int_0^1 x^i A_i(t\mathbf{x})dt\right) = dx^j \int_0^1 \frac{\partial}{\partial x^j}\Big(x^i A_i(t\mathbf{x})\Big)dt \\
&= dx^j \int_0^1 \Big(\delta^i_j A_i(t\mathbf{x}) + x^i t \underbrace{\partial_j A_i(t\mathbf{x})}_{=\partial_i A_j(t\mathbf{x}) \text{ because } dA=0}\Big)dt \\
&= dx^j \int_0^1 \Big(A_j(t\mathbf{x}) + tx^i\partial_i A_j(t\mathbf{x})\Big)dt = dx^j \int_0^1 \frac{d}{dt}\Big(tA_j(t\mathbf{x})\Big)dt \\
&= dx^j\, tA_j(t\mathbf{x})\Big|_{t=0}^{t=1} = A\,,
\end{aligned}$$

therefore $A = df$ is exact, as claimed. ☐

**Remark 6.3.** In the proof for 1-forms above, the function $f$ is nothing but the integral $\int_{[\mathbf{x}_0\mathbf{x}]} A$ over the straight line segment $[\mathbf{x}_0\mathbf{x}]$. A similar proof for $k > 1$, which we skipped, also uses integration, but of course it has to be in a more sophisticated way (since the integral of a $k$-form over a $k$-surface would give a number, not a $(k-1)$-form).

Although the Poincaré lemma is the statement that certain cohomology vanishes, namely, $H^k(\mathbb{R}^n) = 0$ for $k \geq 0$, it plays the key role in calculating cohomology for manifolds when it is non-zero.

The Poincaré lemma is closely related with the fundamental property of de Rham cohomology called 'homotopy invariance'.

**Definition 6.4.** Two smooth maps $f_0, f_1 \colon M \to N$ are *homotopic* if there is a smooth map $F \colon M \times [0,1] \to N$ such that $F(\mathbf{x}, 0) = f_0(\mathbf{x})$ and $F(\mathbf{x}, 1) = f_1(\mathbf{x})$ for all $\mathbf{x} \in M$. Notation: $f_0 \sim f_1$. (In other words, there is a family $f_t = F(\cdot, t)$ giving 'smooth interpolation' between $f_0$ and $f_1$.) The map $F$ is known as a *homotopy* between $f_0$ and $f_1$.

Of course, homotopy also makes sense just for topological spaces, not manifolds. In such a case instead of smoothness we impose continuity. What we have defined above is known as 'smooth homotopy', but we shall skip the adjective.

**Theorem 6.2 (Homotopy invariance).** *Homotopic maps of smooth manifolds induce the same linear map of their de Rham cohomology.*

*Proof.* Suppose $f_0$ and $f_1$ are two homotopic maps $M \to N$. Let $F \colon M \times [0,1] \to N$ be a homotopy. We wish to show that the linear maps $f_0^* \colon H^k(N) \to H^k(M)$ and $f_1^* \colon H^k(N) \to H^k(M)$ coincide for all $k \geq 0$. That means that if we take a cohomology class $[\omega] \in H^k(N)$ represented by a closed $k$-form $\omega \in \Omega^k(N)$, the pull-backs $f_0^*(\omega)$ and $f_1^*(\omega)$ not necessarily coincide, but they must differ only by an exact form. To this end, we shall introduce a linear transformation

$$K \colon \Omega^k(N) \to \Omega^{k-1}(M)$$

decreasing degrees such that for any form $\omega \in \Omega^(N)$,

$$f_1^*\omega - f_\Omega^*\omega = (d \circ K + K \circ d)\omega\,.$$

Thus for closed forms there will be $f_1^*\omega - f_0^*\omega = d(K\omega)$. We define $K$, using the homotopy $F$. Consider $F^*\omega \in \emptyset^k(M \times [0,1])$. Each $k$-form $\sigma$ on $M \times [0,1]$ can be uniquely written as $\sigma = \sigma_t^{(0)} + dt \wedge \sigma_t^{(1)}$ where $\sigma_t^{(0)}$ and $\sigma_t^{(1)}$ are forms on $M$ depending on the parameter $t$. (Use some local coordinates on $M$ to see how it works.) The degree of $\sigma_t^{(0)}$ is $k$, while the degree of $\sigma_t^{(1)}$ is $k-1$. Indeed, the term with $\sigma_t^{(1)}$ contains an extra factor of $dt$, which raises the degree by 1. We define the operation $K$ by $K\omega = (\tilde{K} \circ F^*)\omega$, where $\tilde{K}$ sends any $\sigma = \sigma_t^{(0)} + dt \wedge \sigma_t^{(1)}$ on $M \times [0,1]$ to $\int_0^1 dt(\sigma_t^{(1)})$ (integration over $t$ as a parameter). The result is a $(k-1)$-form. Let us check that $K$ has the desired property. Firstly, $d \circ F^* = F^* \circ d$, so we need to calculate the commutator of $d$ and $\tilde{K}$. On $M \times [0,1]$,

$$d = d_x + dt \wedge \frac{\partial}{\partial t}$$

where $d_x$ can be considered as the differential on $M$. In local coordinates, $d_x = dx^i \wedge \frac{\partial}{\partial x^i}$. If we apply it to $\sigma$ written as above, we arrive at

$$d\sigma = \left(d_x + dt \wedge \frac{\partial}{\partial t}\right)\left(\sigma_t^{(0)} + dt \wedge \sigma_t^{(1)}\right) = d_x\sigma_t^{(0)} + dt \wedge \left(-d_x\sigma_t^{(1)} + \frac{\partial \sigma_t^{(0)}}{\partial t}\right)\,.$$

Hence

$$\tilde{K} d\sigma = \int_0^1 dt \left( -d_x \sigma_t^{(1)} + \frac{\partial \sigma_t^{(0)}}{\partial t} \right) = - \int_0^1 dt \left( d_x \sigma_t^{(1)} \right) + \sigma_1^{(0)} - \sigma_0^{(0)}.$$

On the other hand, clearly

$$d\tilde{K}\sigma = d \int_0^1 dt(\sigma_t^{(1)}) = \int_0^1 dt \left( d_x \sigma_t^{(1)} \right) .$$

Therefore

$$\left( d \circ \tilde{K} + \tilde{K} \circ d \right) \sigma = \sigma_1^{(0)} - \sigma_0^{(0)} .$$

Now we can apply this to $\sigma = F^*\omega$. We have, for $K = \tilde{K} \circ F^*$,

$$\begin{aligned}(d \circ K + K \circ d)\, \omega &= \left( d \circ \tilde{K} \circ F^* + \tilde{K} \circ F^* \circ d \right) \omega \\ &= \left( d \circ \tilde{K} \circ F^* + \tilde{K} \circ d \circ F^* \right) \omega = \left( d \circ \tilde{K} + \tilde{K} \circ d \right) (F^*\omega) \\ &= (F^*\omega)_1^{(0)} - (F^*\omega)_0^{(0)} = f_0^*\omega - f_1^*\omega \, .\end{aligned}$$

Indeed, when we view $F$ as the family $f_t = F(\cdot, t)$, we see that $f_t^*\omega$ is obtained from $F^*\omega$ by setting $dt$ to zero, i.e., by passing to $(F^*\omega)_t^{(0)}$ in the above notation. Hence $f_0^*\omega$ and $f_1^*\omega$ are obtained by additionally setting $t$ to 0 and to 1, respectively, which gives the desired formula. □

An immediate application of this property is as follows.

Two manifolds (more generally, two topological spaces) are *homotopy equivalent* if there are maps in the opposite directions such that their compositions are homotopic to identities. That is, for $X$ and $Y$, there are $f\colon\ X \to Y$ and $g\colon\ Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$. Notation: $X \sim Y$.

**Example 6.6.** $\mathbb{R}^n \sim \{*\}$ (a single point). We can identify the point with $0 \in \mathbb{R}^n$. Consider the obvious maps $i\colon\ \{0\} \to \mathbb{R}^n$ (inclusion) and $p\colon\ \mathbb{R}^n \to \{0\}$ (projection). We have $p \circ i = \mathrm{id}$, but $i \circ p$ is not the identity, it is the map that send every vector to zero. However, it is homotopic to the identity, the family $f_t\colon\ \mathbf{x} \mapsto t\mathbf{x}$, where $t \in [0, 1]$, giving the desired homotopy.

**Example 6.7.** Both open and closed cylinders, $S^1 \times \mathbb{R}$ and $S^1 \times [-1, 1]$, are homotopy equivalent to the circle $S^1$. (The same is true if we replace $S^1$ by any space $X$.) (Construct the maps and homotopies following the previous example.)

**Corollary 6.2.** *Homotopy equivalent manifolds $M$ and $N$ have isomorphic cohomology groups $H^k(M)$ and $H^k(N)$ for all $k$.*

*Proof.* Let $f\colon M \to N$ and $g\colon N \to M$ be such maps that $f \circ g \sim \mathrm{id}_N$ and $g \circ f \sim \mathrm{id}_M$. Then by Theorem 6.2, for each $k$, we have linear maps of vector spaces $f^*\colon H^k(N) \to H^k(M)$ and $g^*\colon H^k(M) \to H^k(N)$ with the property $f^* \circ g^* = \mathrm{id}$ on $H^k(M)$ and $g^* \circ f^* = \mathrm{id}$ on $H^k(N)$ ('homotopic to identity' becomes 'equal to identity' for the maps of cohomology). That mean that these vector spaces are isomorphic. □

From here it follows that $\mathbb{R}^n$ (or any star-shaped domain) being homotopy equivalent to a point, has the same cohomology as a point. Obviously, it implies that $H^0(\mathbb{R}^n) \cong \mathbb{R}$ and $H^k(\mathbb{R}^n) = 0$ for $k > 0$. We again arrive at the Poincaré lemma.

## 6.3 $n$-th de Rham cohomology of $n$-dimensional compact manifold

**Theorem 6.3.** *Let $M$ be $n$-dimensional smooth orientable compact manifold. Then $H^n_{DR}(M) = \mathbb{R}^n$.*

Proof of the Theorem.

Let $M$ be smooth compact orientable manifold.

Consider volume form on $M$. It is closed form. $\int_M \sigma \neq 0$, Hence according to Stokes theorem $\sigma$ is not exact form. (If $\sigma = dr$ then $\int_M \sigma = \int_M \sigma = \int_M dr = \int_{\partial M} r = 0$)

Hence we see that equivalence class of the form $\sigma$ is not equal to zero: $[\sigma] \neq 0$. Hence

$$H^n_{DR}(M) \neq 0$$

To show that $H^n_{DR}(M) \neq \mathbb{R}$ we use the following lemma:

**Lemma** : $n$-form $\theta$ on $M$ is exact if and only if $\int_M \theta = 0$

Indeed let $[\omega]$ be an arbitrary cohomological class $[w] \in H^n_{DR}(M)$. Let $\int_M \omega = k\sigma$. One can see that the form $\omega' = \omega - k\sigma$ obeys the conditions of the lemma and is exact. Hence $[\omega] = k[\sigma]$. Thus we prove that all elements in $H^n_{DR}(M)$ are proportional to $[\sigma]$.

It remains to prove the Lemma. This lemma follows from the following three propositions:

**Proposition 6.3.** *If $n$-form is compactly supported in $\mathbb{R}^n$ and $\int_M \omega = 0$ then $\omega = d\theta$ such that $\theta$ is compactly supported too.*

**Proposition 6.4.** *Let two coordinate domains domains $U_1, U_2$ intersect by domain $W = U_1 \cap U_2$ Let $n$-form $\omega_1$ is supported in the coordinate domain $U_1$, and $n$-form $\omega_2$ is supported in the coordinate domain $U_2$ and*

$$\int_M \omega_1 = \int_M \omega_2$$

*Then there exists $n$-form $\rho$ such that $\rho$ is supported in the domain $W = U_1 \cap U_2$ and*

$$\int_M \rho = \int_M \omega_1 = \int_M \omega_2 \tag{6.1}$$

*(We suppose that orientation on the domains $U_1, U_2$ is chosen)*

**Proposition 6.5.** *Let two coordinate domains domains $U_1, U_2$ intersect by domain $W = U_1 \cap U_2$ Let $n$-form $\omega_1$ is supported in the coordinate domain $U_1$, and $n$-form $\omega_2$ is supported in the coordinate domain $U_2$ and*

$$\int_M \omega_1 = \int_M \omega_2$$

*Then form $\omega_1 - \omega_2$ is exact.*

The first Propostition is almost obvious. Suppose for example that $\omega = a(x,y)dx \wedge dy$ has support in the $0 \leq x \leq 1, 0 \leq y \leq 1$. Then consider

$$\theta(x,y)dy = \left[\int_{-\infty}^{x} a(t,y)dt\right] dy$$

One can see that $\theta(x,y)$ vanishes for $(x,y) \notin [0,] \times [0,1]$ because $\int \omega = 0$.

To prove the next proposition we just consider a bump function $f_W$ associated with domain $W$. Let $y^1, y^2 \ldots, \ldots, y^n$ be an arbitrary local coordinates in $W$ then consider an $n$-form

$$\rho = c f_W dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n dy^n$$

where one choose constant $c$ such that condition (6.1) fulfills.

Now we prove the last proposition. Consider the form $\rho$ such that its support is in the intersection of the domains $U_1, U_2$ and

$$\int_M \rho = \int_M \omega_1 = \int_M \omega_2 \tag{6.2}$$

Then we see that the forms $\omega_1$ and $\rho$ have support in the domain $U_1$ and the forms $\omega_2$ and $\rho$ have support in the domain $U_2$. Then due to (6.2) $\int(\omega_1 - \rho) = 0$ and $\int(\omega_2 - \rho) = 0$ hence due to Proposition6.3

$$\omega_1 - \rho = d\theta_1, \qquad \rho - \omega_2 = d\theta_2 \Rightarrow \omega_1 = \omega_2 + d(\theta_1 + \theta_2)$$

Now take an arbitrary but fixed small enough coordinate domain $U_0$ and an arbitrary form $\sigma$ such that this form has a support in this domain and $\int \sigma = 1$. Then one can see that if $U$ is an arbitrary coordinate domain and $\omega$ is an arbitrary form such that this form has a support in the domain $U$ then

$$\omega - k\sigma = d\theta, \text{ where } k = \int_M \omega \tag{6.3}$$

Indeed if $k = 0$ this is just the statement of the proposition6.3. If domains $U_0$ and $U$ intersect it is the statement of the last proposition. In the general case one have consider the chain of the intersected domains.

Using these Propositions and partition of unity one can prove the Lemma.

Indeed let $\{U_a, \varphi_\alpha\}$ be finite oriented cover and let $\{\rho_\alpha\}$ be partition of unity subordinate to this cover. Let $\omega$ be $n$-form on manifold such that $\int \omega = 0$. Consider

$$\omega = 1 \cdot \omega = \sum \rho_\alpha \omega = \sum \omega_\alpha, \text{ where } \omega_a = \rho_\alpha \omega$$

Now the statement of lemma follows from the (6.3). Indeed

$$[\omega_\alpha] = k_\alpha \sigma \text{ where } k = \int_M \omega_\alpha \Rightarrow [\omega] = \sum [\omega_\alpha] = [\sigma] \sum k_\alpha = \int_M \sum \omega_\alpha = 0$$

The lemma is proved.